Continuum Mass and Force Concepts

Introduce the notion for a continuum body and the various types of forces acting on it. A continuum is is infinitely divisible, and hence we ignore the atomic nature of materials. At length scales much larger than atomic spacing this leads to effective models.

The discussion of the internal forces leads to notion of stress tensor field. We introduce mechanical equilibrium and corresponding differential equations.

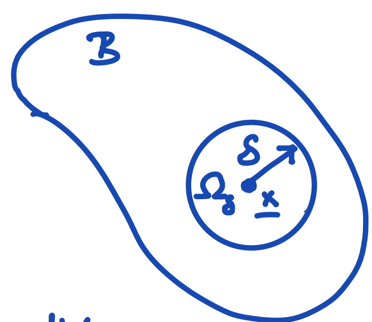
Important ideas:

1) Notion of mass density field
2) Notion of body and surface forces
3) Cauchy - stress field
4) Equations of equilibrium

Mass Density

Mass is a physical property of matter that quantifies its resistance to acceleration when a force is applied. In the continuum assumption we assume that mass is continuously distributed through out the volume of a body, $\mathcal{B}$.

We assume that any subset Ω of $\mathcal{B}$ with positive volume has a positive mass.



$$V_\Omega = \int_\Omega dV \qquad m_\Omega = \int_\Omega \rho(\underline{x})\, dV$$

where $\rho(\underline{x})$ is the mass density field, which can be defined at any point $\underline{x}$ as

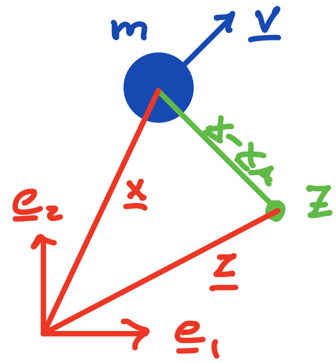
$$\boxed{\rho(\underline{x}) = \lim_{\delta \to 0} m_{\Omega_\delta} / V_{\Omega_\delta}}$$

The center of volume (centroid) and mass are

$$\underline{x}_v = \frac{1}{V_\Omega} \int_\Omega \underline{x}\, dV \quad \text{and} \quad \underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \underline{x}\, \rho(\underline{x})\, dV$$

Short review of force and moment

Object with a mass m and velocity $\underline{v}$ has a momentum:



Linear momentum: $\underline{L} = m\,\underline{v}$

angular momentum: $\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$

$\rightarrow$ always relative to a point!

Newtons 1^{st} law: "Principle of inertia"

In a fixed frame of reference every object preserves its state of motion unless it is acted upon by a force or torque.

Force: $\underline{f} = \dfrac{d\underline{L}}{dt} = \dfrac{d(m\underline{v})}{dt} = m\dfrac{d\underline{v}}{dt} = m\,\underline{a}$ $\qquad \dfrac{d\underline{v}}{dt} = \dot{\underline{v}}$

$\underline{f} = m\,\underline{a}$ $\qquad$ Newton's 2^{nd} law

Torque: $\underline{\tau} = \dfrac{d\underline{j}}{dt} = m\dfrac{d}{dt}(\underline{x} \times \underline{v} - \underline{z} \times \underline{v}) =$

$= m(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}} - \cancel{\dot{\underline{z}}}^{\,0} \times \underline{v} - \underline{z} \times \dot{\underline{v}})$

$= m(\cancel{\underline{v} \times \underline{v}}^{\,0} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}) = m(x - \underline{z}) \times \dot{\underline{a}}$

Body Forces

The interactions between parts of a body or a body and its environment are described by forces. Any force that not due to physical contact is a body force and acts on the entire body. Common body forces originate from gravitational and electromagnetic field.

If $\underline{b}$ is a body force field with units $\frac{\text{force}}{\text{volume}}$ the the resultant force on a body is

$$\underline{F}_b = \int_\Omega \underline{b}(\underline{x})\, dV$$

and the torque on a body about a point $\underline{z}$ is given by
$$\underline{T}_b = \int_\Omega (\underline{x} - \underline{z}) \times \underline{b}(\underline{x})\, dV$$

Units: $\frac{F}{V} = \frac{a\,m}{V}$ $\left[\frac{1}{L^3}\frac{L}{T^2}M = \frac{M}{L^2T^2}\right]$

Example: gravitational body force

$$\boxed{\underline{b}_g = \rho g} \qquad \left[\frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}\right]$$

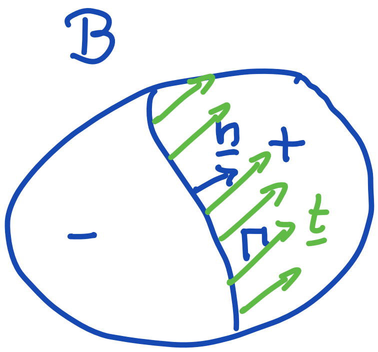
Inertial/fictitious forces in rotating frames such as the centrifugal and the Coriolis force act similar to body forces.

Surface/Contact Forces

arise due to the physical contact between bodies. Forces along imaginary surfaces within a body are called internal forces while forces along the bounding surface of a body are external.

Internal surface forces hold a body together. External surface forces describe the interaction with the environment.

Traction Field



Consider an arbitrary surface Γ in B with unit normal $\underline{n}(\underline{x})$ that defines the positive and negative sides of B.

The force per unit area exerted by material on the pos. side upon material on the neg. side is given by the traction field $\underline{t}_n$ for Γ.

The resultant force due to a traction field on Γ is

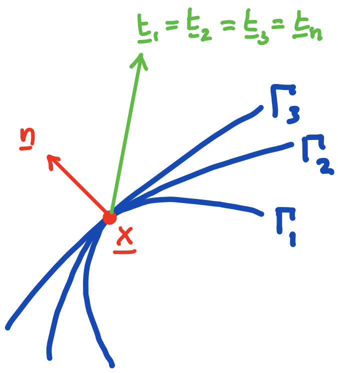
$$\boxed{\underline{r}_s[\Gamma] = \int_\Gamma \underline{t}_n(\underline{x})\, dA}$$

The resultant torque about point $\underline{z}$ due to a traction field on Γ is

$$\boxed{\underline{\tau}_s[\Gamma] = \int_\Gamma (\underline{x}-\underline{z}) \times \underline{t}_n(\underline{x})\, dA}$$

Cauchy's postulate

The traction field $\underline{t}_n$ on a surface Γ in B depends only pointwise on the unit normal field $\underline{n}$. In particular, there is a traction function such that $\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}), \underline{x})}$.



This assumes that the traction field is independent of $\nabla \underline{n}$ and hence the curvature of the surface. Therefore the traction $\underline{t}_i$ on the set of surfaces Γ_i that are tangent at $\underline{x}$ is the same, $\underline{t}_i = \underline{t}_n$.

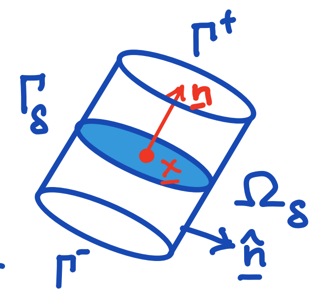
Law of Action and Reaction

If the traction field, $\underline{t}(\underline{n},\underline{x})$, is continuous and bounded, then

$$\boxed{\underline{t}(-\underline{n},\underline{x}) = -\underline{t}(\underline{n},\underline{x})}$$

for all $\underline{n}$ and $\underline{x} \in B$.

To show this consider a disk D with arbitrary fixed radius around $\underline{x}$. Let Ω_δ be the cylinder with center $\underline{x}$, axis $\underline{n}$ and height $\delta > 0$.



We refer to the end-faces of the cylinder as as Γ^+ and Γ^- and the mantle as Γ_δ. Let $\hat{\underline{n}}$ be the outward normal on $\partial\Omega_\delta$.

Note: on Γ^+ $\hat{\underline{n}} = \underline{n}$ and on Γ^- $\hat{\underline{n}} = -\underline{n}$

Also as $\delta \to 0$ the area of the end faces approach the disk, $\Gamma^{\pm} \to D$ but $\Gamma_\delta \to 0$.

Since $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma^+ \cup \Gamma^-$ we have

$$\lim_{\delta\to 0}\left[\int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}}(y,y)\, dA + \int_{\Gamma^+} \underline{t}(\underline{n},\underline{y})\, dA + \int_{\Gamma^-} \underline{t}(-n,\underline{y})\, dA\right] = 0$$

the first term vanishes becaus $\underline{t}$ is bounded and $\Gamma_\delta \to 0$. Using the fact that $\Gamma^\pm \to D$ we have in the limit

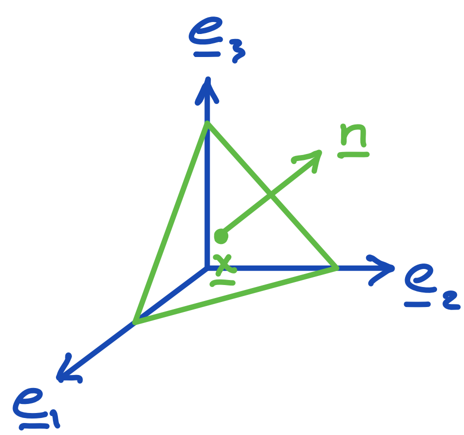
$$\int_D \underline{t}(\underline{n},\underline{y}) + t(-\underline{n},y)\, dA = 0$$

since the radius of D is arbitrary the integrand must vanish so that $\underline{t}(\underline{n},\underline{x}) + \underline{t}(-\underline{n},\underline{x}) = 0$ ✓

The Stress tensor

Cauchy's Theorem

Let $\underline{t}(\underline{n},\underline{x})$ be the traction field for body B that satisfies Cauchy's postulate. Then $\underline{t}(\underline{n},\underline{x})$ is linear in $\underline{n}$, that is, for each $\underline{x} \in B$ there is a second-order tensor field $\underline{\underline{\sigma}}(\underline{x}) \in V^2$ such that $\boxed{\underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$
called the Cauchy stress field for B.



To estabish this consider a frame $\{\underline{e}_i\}$, a point $\underline{x} \in B$ and a normal $\underline{n}$ s.t. $\underline{n} \cdot \underline{e}_i > 0$.

For $\delta > 0$, let Γ_δ denote a triangular region with center $\underline{x}$, normal $\underline{n}$ and maximum edge length δ.

Let Ω_δ be the tetrahedron bounded by Γ_δ and the three coordinate planes. These planes form three faces Γ_j with outward normals $\underline{n}_j = -\underline{e}_j$. The volume of Ω_δ goes to zero as δ becomes small.

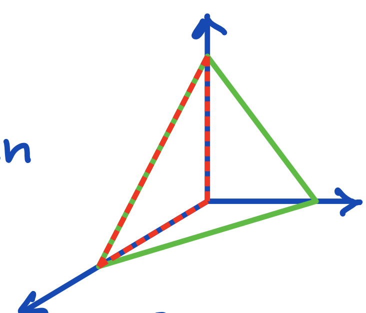
$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} \underline{t}(\underline{n}(\underline{y}), \underline{y})\, dA = 0$$

where $A_{\partial\Omega_\delta}$ is the surface area of Ω_δ

Since $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we have

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j, \underline{y})\, dA \right] = 0$$

Since each face Γ_j can be linearly mapped onto Γ_δ with constant Jacobian



$n_j = \underline{n} \cdot \underline{e}_j > 0$ so that $A_{\Gamma_j} = n_j A_{\Gamma_\delta}$

$\Rightarrow A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=1}^{3} \Gamma_j = \lambda A_{\Gamma_\delta}$ $\qquad \lambda = 1 + \sum_{j=1}^{3} n_j$

substituting we obtain

$$\lim_{\delta\to 0} \frac{1}{A_{\partial\Omega_\delta}}\left[\int_{\Gamma_\delta} \underline{t}(\underline{n},\underline{y})\, dA + \sum_{j=1}^{3} \int_{\delta} t_n(-\underline{e}_j,\underline{y})\, n_j\, dA\right] = 0$$

$$\lim_{\delta\to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \underline{t}(\underline{n},\underline{y}) + \sum_{j=1}^{3} t(-\underline{e}_j,\underline{y})\, n_j\, dA = 0$$

As $\delta \to 0$ the area Γ_δ shrinks to $\underline{x}$ so that by the mean value Thm for integrals the limit is given by the integrand. Hence

$$\underline{t}(\underline{n},\underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\, n_j = 0$$

Using the Law of Action and Reaction

$$\underline{t}(\underline{n},\underline{x}) = -\sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\, n_j = \sum_{j=1}^{3} \underline{t}(\underline{e}_j,\underline{x})\, n_j$$

or with summation convention

$$\underline{t}(\underline{n},\underline{x}) = \underline{t}(\underline{e}_j,\underline{x})\, n_j$$

using the definition of dyadic product

$$\left(\underline{t}(\underline{e}_j,\underline{x})\otimes\underline{e}_j\right)\underline{n} = \underbrace{(\underline{e}_j\cdot\underline{n})}_{n_i\,\underline{e}_j\cdot\underline{e}_i \,=\, n_i\,\delta_{ij} \,=\, n_j}\,\underline{t}(\underline{e}_j,\underline{x})$$

So that we have

$$\underline{t}(\underline{n},\underline{x}) = \left(\underline{t}(\underline{e}_j,\underline{x})\otimes\underline{e}_j\right)\underline{n} = \underline{\underline{\sigma}}\,\underline{n}$$

$$\underline{\underline{\sigma}} = \underline{t}(\underline{e}_j,\underline{x})\otimes\underline{e}_j$$

substituting $\underline{t}(\underline{e}_j,\underline{x}) = t_i(\underline{e}_j,\underline{x})\,\underline{e}_i$ we obtain the definition of the Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\,\underline{e}_i\otimes\underline{e}_j \quad \text{with} \quad \sigma_{ij} = t_i(\underline{e}_j,\underline{x})}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.

The traction vectors on the coor. planes at $\underline{x}$ are

$$\underline{t}(\underline{e}_1,\underline{x}) = t_i(\underline{e}_1,\underline{x})\,\underline{e}_i = S_{i1}(\underline{x})\,\underline{e}_i$$

$$\underline{t}(\underline{e}_2,\underline{x}) = t_i(\underline{e}_2,\underline{x})\,\underline{e}_i = S_{i2}(\underline{x})\,\underline{e}_i$$

$$\underline{t}(\underline{e}_3,\underline{x}) = t_i(\underline{e}_3,\underline{x})\,\underline{e}_i = S_{i3}(\underline{x})\,\underline{e}_i$$