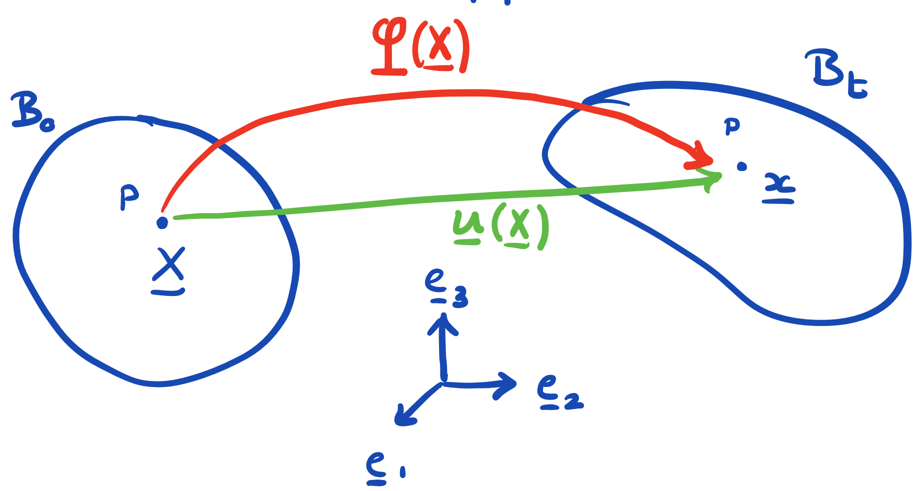
Kinematics

Study of geometry of motion without consideration of mass or stress.

⇒ Quantify the strain and rate of strain.

Deformation Mapping



B_o = body in reference, initial, undeformed or material configuration

B_t = body in current, spatial or deformed config.

p = material point in body

$\underline{X}$ = location of p in B_o

$\underline{x}$ = location of p in B_t

$\underline{\varphi}(\underline{X})$ = deformation mapping

$\underline{u}(\underline{X})$ = displacement

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ = frame

$\underline{X} = X_I \, \underline{e}_I$ $\quad\quad$ X_I = components of $\underline{X}$ in $\{\underline{e}_I\}$

$\underline{x} = x_i \, \underline{e}_i$ $\quad\quad$ x_i = " " $\underline{x}$ in $\{\underline{e}_i\}$

Convention:

Upper case quantities & indices → reference. B_0

Lower case quantities & indices → current. B_t

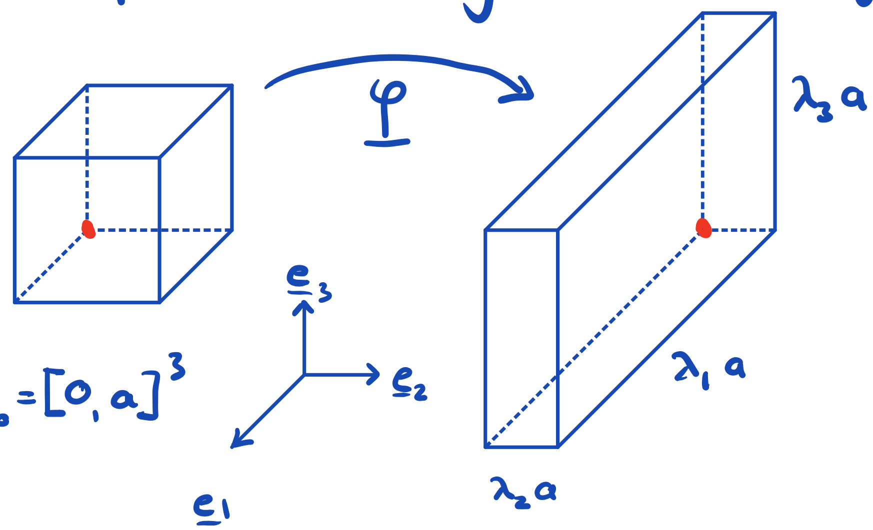
Definition of deformation mapping

$$\boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \varphi_i(\underline{X})\, \underline{e}_i}$$

Displacement of a material particle

$$\boxed{\underline{u}(\underline{X}) = \underline{\varphi}(\underline{X}) - \underline{X}}$$

Example: Streching cube with edge length a



$\mathcal{B}_0 = [0, a]^3$

deformation map:

$$x_1 = \lambda_1 X_1 + v_1$$

$$x_2 = \lambda_2 X_2 + v_2$$

$$x_3 = \lambda_3 X_3 + v_3$$

λ = strech ratio

$\underline{v}$ = translation (only important in presence of body force)

$(\underline{v} = 0)$

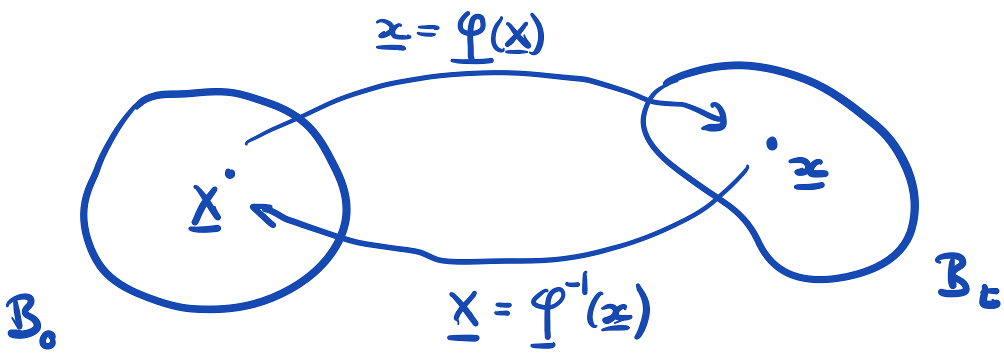
$$\underline{x} = \underline{\varphi}(\underline{X}) = \lambda_1 X_1 \underline{e}_1 + \lambda_2 X_2 \underline{e}_2 + \lambda_3 X_3 \underline{e}_3 = \Lambda_{ij} X_j \underline{e}_i$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\underline{x} = \underline{\underline{\Lambda}}\, \underline{X}$$

Inverse Mapping

If φ is admissible $\Rightarrow$ well defined inverse φ^{-1}

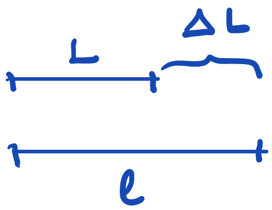


inverse deformation map : $\boxed{\underline{X} = \underline{\varphi}^{-1}(\underline{x})}$

Measures of Strain

In 1D we have simple measures

original:
deformed:



$\Delta L = l - L$

engineering strain: $e = \dfrac{\Delta L}{L} = \dfrac{l - L}{L}$

strech ratio: $\lambda = \dfrac{l}{L}$ $\Rightarrow$ $e = \lambda - 1$

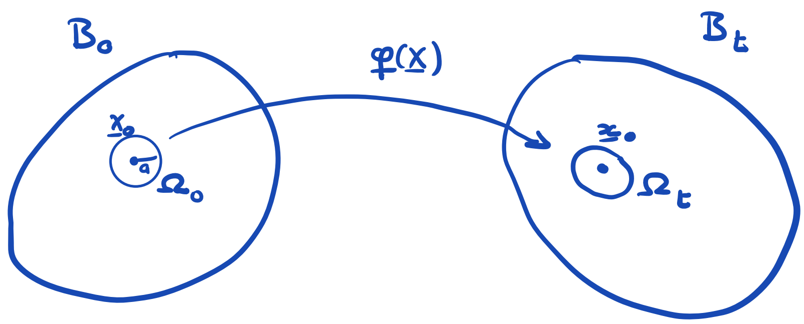
true or Hencky strain: $\varepsilon = \ln(\lambda)$

Green strain: $\varepsilon = \frac{1}{2}(\lambda^2 - 1)$

....

Description of strain is not unique !!

Here we need to find a general 3D approach that is not limited to small deformations.



Sphere Ω_0 of radius a around $\underline{X}_0$.

Mapped to Ω_t around $\underline{x}_0$ by $\varphi(\underline{X})$

$\Omega_t = \{\underline{x} \in B_t \mid \underline{x} = \varphi(\underline{X}),\ \underline{X} \in \Omega_0\} \rightarrow \Omega_t = \varphi(\Omega_0)$

Def: The strain at $\underline{X}_0$ is any relative difference between Ω_0 and Ω_t in limit of $a \rightarrow 0$.

Deformation gradient

Natural way to quantify local strain

$$\boxed{\underline{\underline{F}}(\underline{X}) = \nabla \varphi(\underline{X})} \qquad F_{iJ} = \frac{\partial \varphi_i}{\partial X_J}$$

Expanding deformation in Taylor series around $\underline{X}_0$ we have

$$\varphi(\underline{X}) = \varphi(\underline{X}_0) + \nabla\varphi(\underline{X}_0)(\underline{X} - \underline{X}_0) + \mathcal{O}(|\underline{X} - \underline{X}_0|^2)$$

$$= \underbrace{\varphi(\underline{X}_0) - \nabla\varphi(\underline{X}_0)\underline{X}_0}_{\underline{c}} + \underbrace{\nabla\varphi(\underline{X}_0)}_{\underline{\underline{F}}(\underline{X}_0)}\underline{X}$$

locally we can approximat φ as

$$\boxed{\varphi(\underline{X}) \simeq \underline{c} + \underline{\underline{F}}(\underline{X}_0)\underline{X}} \qquad \text{(affine deform.)}$$

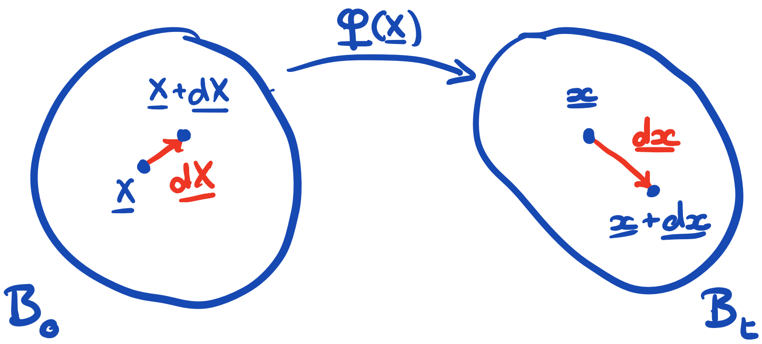
$\Rightarrow$ $\underline{\underline{F}}(\underline{X}_0)$ characterizes <u>local</u> behavior of $\varphi(\underline{X})$

Homogeneous deformation

$\underline{\underline{F}}$ is constant

$$\Rightarrow \quad \boxed{\underline{x} = \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}}$$

Consider the mapping of line segment



$|d\underline{X}| \ll 1$

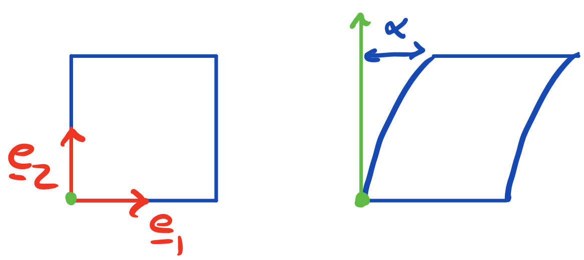
$|d\underline{x}| \ll 1$

$$\underline{x} + d\underline{x} = \underline{\varphi}(\underline{X}+d\underline{X}) \simeq \varphi(\underline{X}) + \nabla\underline{\varphi}(\underline{X})\, d\underline{X} = \underline{x} + \underline{\underline{F}}(\underline{X})\, d\underline{X}$$

$$\boxed{\begin{aligned} d\underline{x} &= \underline{\underline{F}}(\underline{X})\, d\underline{X} \\ dx_i &= F_{iJ}(\underline{X})\, dX_J \end{aligned}}$$

$\underline{\underline{F}}$ maps material vectors into spatial vectors.

Example: Shear deformation



$$\underline{\varphi}(\underline{X}) = [X_1 + \alpha X_2^2,\ X_2]$$

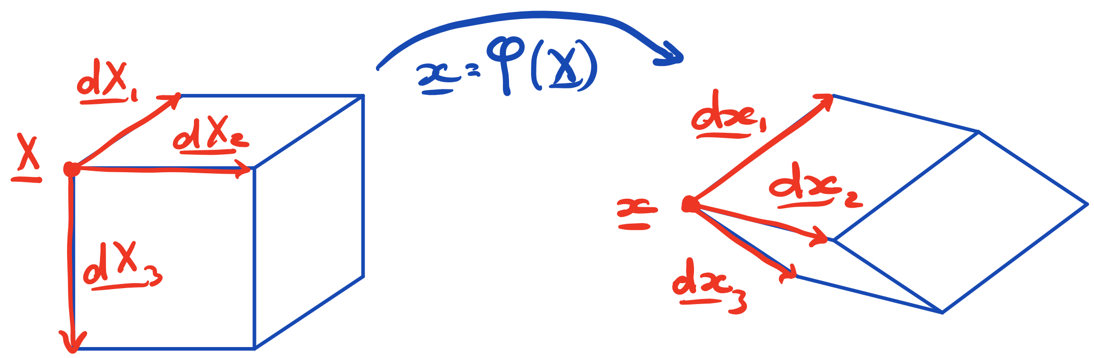
$$\nabla\varphi = \underline{\underline{F}} = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix}$$

$\underline{\underline{F}}\,\underline{e}_1 = [1, 0]^T = \underline{e}_1$ unchanged

$\underline{\underline{F}}\,\underline{e}_2 = [2\alpha X_2, 1]^T$ rotated and streched

Volume changes

Change in volume during deformation



Volumes are: $dV_X = (d\underline{X}_1 \times d\underline{X}_2) \cdot d\underline{X}_3$

$$dV_x = (d\underline{x}_1 \times d\underline{x}_2) \cdot d\underline{x}_3$$
$$= \det([d\underline{x}_1][d\underline{x}_2][d\underline{x}_3])$$

substituting $d\underline{x} = \underline{\underline{F}}\, d\underline{X}$

$$dV_x = \det([\underline{\underline{F}}\, d\underline{X}_1][\underline{\underline{F}}\, d\underline{X}_2][\underline{\underline{F}}\, d\underline{X}_3])$$
$$= \det(\underline{\underline{F}}\, d\underline{\underline{X}}) \quad \text{where} \quad d\underline{\underline{X}} = [d\underline{X}_1\ d\underline{X}_2\ d\underline{X}_3]$$
$$= \det(\underline{\underline{F}}) \det(d\underline{\underline{X}})$$
$$= \det(\underline{\underline{F}}) (d\underline{X}_1 \times d\underline{X}_2) \cdot d\underline{X}_3$$

$$\Rightarrow \quad \boxed{dV_x = \det(\underline{\underline{F}})\, dV_X}$$

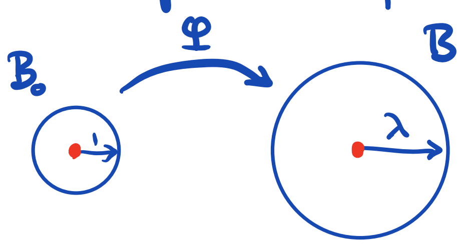
The field $J(\underline{X}) = \det(\underline{\underline{F}}) = \dfrac{dV_x}{dV_{\underline{X}}}$ is the Jacobian of φ and measures the volume strain.

$J(\underline{X}) > 1$: volume increase

$J(\underline{X}) < 1$: volume decrease

$J(\underline{X}) = 1$: no volume change

Example: Expanding shere $V = \frac{4}{3}\pi R^3$



$$V_0 = \frac{4\pi}{3}$$

$$V_t = \frac{4\pi}{3}\lambda^3$$

Deformation map: $\underline{\underline{x}} = \varphi(\underline{X}) = \lambda \underline{X}$ $\qquad \lambda > 1$

$$\underline{\underline{F}} = \nabla\varphi = \lambda \underline{\underline{I}}$$

$J \neq J(\underline{X})$ because $\underline{\underline{F}}$ is const

$$J = \det(\underline{\underline{F}}) = \det(\lambda \underline{\underline{I}}) = \lambda^3 \underbrace{\det(\underline{\underline{I}})}_{1}$$

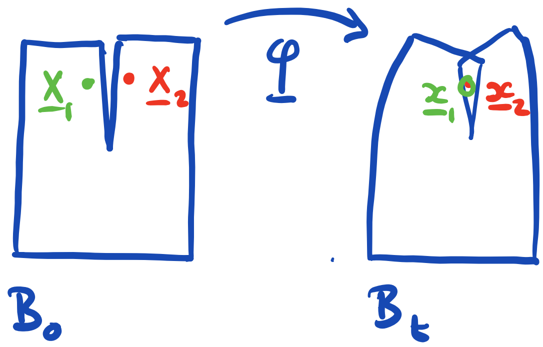
$$J = \lambda_3$$

$$V_t = J\, V_0 = \frac{4\pi}{3}\lambda^3 \checkmark$$

Admissible deformations

For φ to represent the deformation of a body it must satisfy the following conditions:

1) $\underline{\varphi} : B_0 \to B_t$ is one to one and onto



two separate points in B_0 cannot be mapped to same point in B_t.

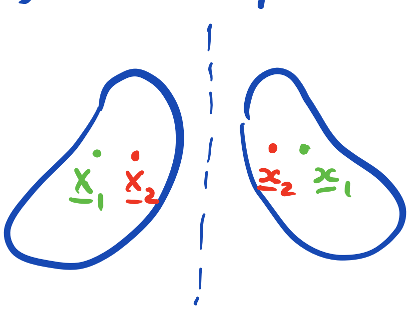
one to one: for each $\underline{X}$ in B_0 there is at most one

$\underline{x}$ in B_t s.t. $\underline{x} = \underline{\varphi}(\underline{X})$

onto: for each $\underline{X}$ in B_0 there is at least one

$\underline{x}$ in B_t s.t. $\underline{x} = \underline{\varphi}(\underline{X})$

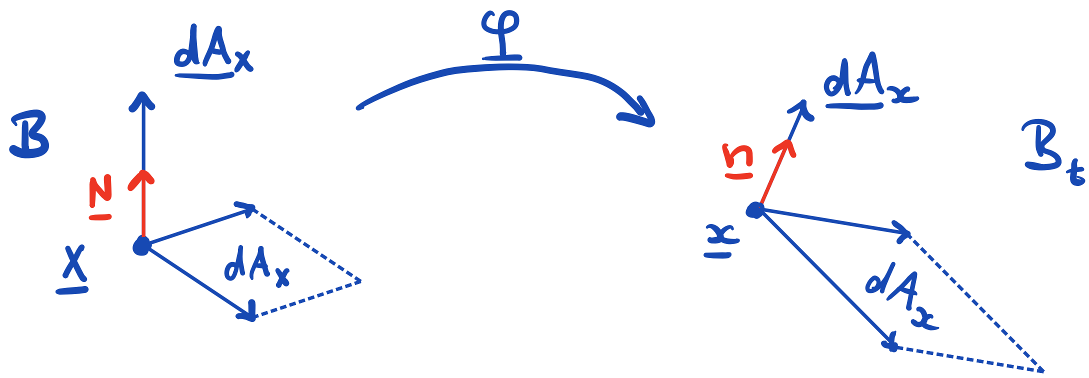
2) $\det(\nabla \underline{\varphi}) > 0$



The orientation of a body is preserved, i.e., a body cannot be deformed into its mirror image.

Surface area changes

How do surfaces change during deformation



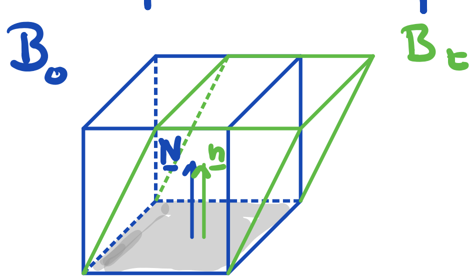
surface normals: $|\underline{N}| = |\underline{n}| = 1$

surface vector elements: $d\underline{A}_X = \underline{N}\, dA_X$

$d\underline{A}_x = \underline{n}\, dA_x$

Important: $\underline{n} \neq \underline{\underline{F}}\,\underline{N}$!

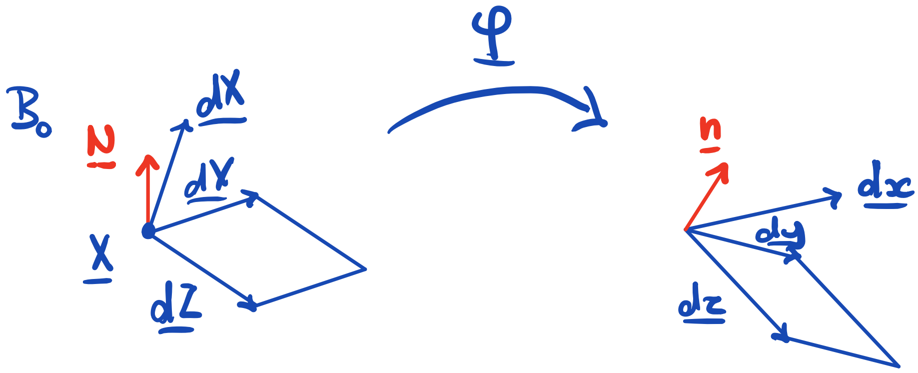
Example: Simple shear



$\underline{N} = \underline{n} \neq \underline{\underline{F}}\,\underline{N}$

What is the relation between $\underline{N}$ and $\underline{n}$?

Consider $\underline{dX}$ so that $\underline{N} \cdot \underline{dX} \neq 0$



$$\underline{dA}_X = \underline{dY} \times \underline{dZ} \qquad \underline{dA}_x = \underline{dy} \times \underline{dz}$$

$$dV_X = \underline{dA}_X \cdot \underline{dX} \qquad dV_x = \underline{dA}_x \cdot \underline{dx}$$

Change in volume: $dV_x = J \, dV_X$

$$\underline{dA}_x \cdot \underline{dx} = J \, \underline{dA}_X \cdot \underline{dX} \qquad \text{with} \quad \underline{dx} = \underline{\underline{F}} \, \underline{dX}$$

$$\underline{dA}_x \cdot \underline{\underline{F}} \, \underline{dX} - J \, \underline{dA}_X \cdot \underline{dX} = 0 \qquad \text{using transpose}$$

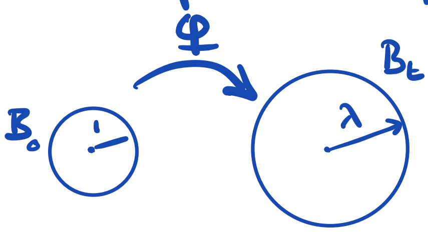
$$\underline{\underline{F}}^T \underline{dA}_x \cdot \underline{dX} - J \, \underline{dA}_X \cdot \underline{dX} = 0$$

$$\left(\underline{\underline{F}}^T \underline{dA}_x - J \, \underline{dA}_X \right) \cdot \underline{dX} = 0 \qquad \text{since } \underline{dX} \text{ is arbitrary}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \underline{dA}_x &= J \, \underline{\underline{F}}^{-T} \underline{dA}_X \\ \underline{n} \, dA_x &= J \, \underline{\underline{F}}^{-T} \underline{N} \, dA_X \end{aligned}} \qquad \text{Nanson's formula}$$

so that $$\underline{n} = \underbrace{\frac{J\, dA_X}{dA_x}}_{\text{normalization}} \quad \underbrace{\underline{\underline{F}}^{-T}\underline{N}}_{\text{direction}}$$

Example : Expanding shere



$$A_0 = 4\pi \qquad A_t = 4\pi\lambda^2$$

$$A_t / A_0 = \lambda^2$$

$$\underline{x} = \varphi(\underline{X}) = \lambda \underline{X} \qquad \underline{\underline{F}} = \lambda \underline{\underline{I}}$$

$$J = \det(\underline{\underline{F}}) = \lambda^3 \qquad \underline{\underline{F}}^{-T} = \underline{\underline{F}}^{-1} = \frac{1}{\lambda}\underline{\underline{I}}$$

Nansen formula: $\underline{n}\, dA_x = J\, \underline{\underline{F}}^{-T}\underline{N}\, dA_X$

$$J\, \underline{\underline{F}}^{-T}\underline{N} = \lambda^3 \frac{1}{\lambda} \underline{\underline{I}}\,\underline{N} = \lambda^2 \underline{N} \Rightarrow \underline{n}\frac{dA_x}{dA_X} = \lambda^2 \underline{N}$$

taking abs. value : $\frac{dA_x}{dA_X} = \lambda^2$ ✓

Next time: Analysis of local deformation

series of decompositions

I) Translation - Fixed point decomposition

$\varphi(\underline{X}) \longrightarrow$ translation & def. with fixed point

II) Polar decomposition

def with fixed point $\longrightarrow$ rotation & strech

III) Spectral decomposition

strech $\rightarrow$ principal streches

$\Rightarrow$ allows us to formulate strain tensors