Lecture 1: Vectors & Index notation

I, Review of Vectors

Def: Vector is a quantity with a magnitude & direction

$$\underline{v} = |\underline{v}|\, \hat{\underline{v}}$$

$|\underline{v}|$ = magnitude $\quad (|\underline{v}| \geq 0)$

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$ direction $\quad (|\hat{\underline{v}}| = 1) \quad$ unit vector

Examples: force, velocities, displacements, ...

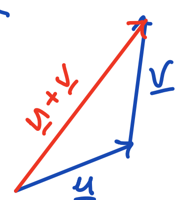
Q: Is it possible to have vector without direction?

$\underline{0}$

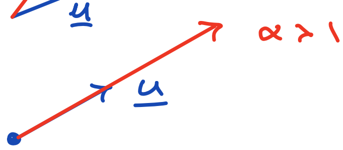
Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication.

$\underline{u} \in \mathcal{V} \quad \underline{v} \in \mathcal{V} \quad \alpha \in \mathbb{R}$

1) $\underline{u} + \underline{v} \in \mathcal{V}$

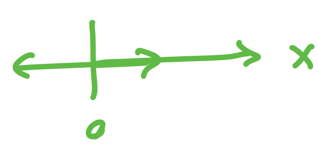


2) $\alpha\, \underline{u} \in \mathcal{V}$

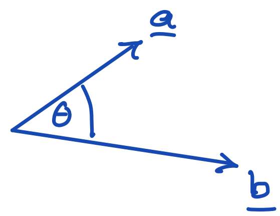


Q1: Do vectors in $\mathbb{R}^3$ form vector space?

Q2: Do vectors in $\mathbb{R}^+$ form vector space?



Scalar product: $\underline{a}, \underline{b} \in \mathcal{V}$



$$\boxed{\underline{a} \bullet \underline{b} = |\underline{a}||\underline{b}| \cos\theta} \qquad \theta \in [0, \pi]$$

$\underline{a} \cdot \underline{b} = 0 \qquad \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

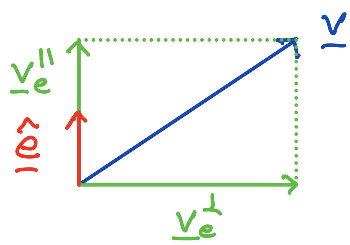
$\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$ commutative

Projection: $\hat{\underline{e}}$ unit vector & $\underline{v} \in \mathcal{V}$

$\underline{V} = \underline{v}_e^{\parallel} + \underline{v}_e^{\perp}$

$\underline{v}_e^{\parallel} = \underline{v} \cdot \underline{e}$

$\underline{v}_e^{\perp} = \underline{v} - \underline{v}_e^{\parallel}$

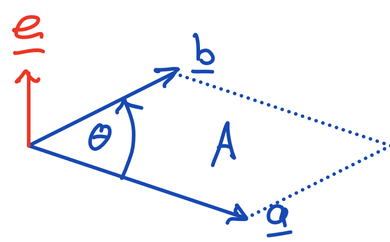


Vector product: $\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta\, \hat{\underline{e}}} \qquad \theta \in [0, \pi]$$

$\hat{\underline{e}}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

direction right-hand rule



$|\underline{a} \times \underline{b}|$ = Area of paralelogram spanned by $\underline{a}$ & $\underline{b}$

Note: $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$ not commutative

Q: What does it mean when $\underline{a} \times \underline{b} = \underline{0}$?

$(\underline{a} \neq \underline{0}, \ \underline{b} \neq \underline{0})$ ~~point~~ ; parallel

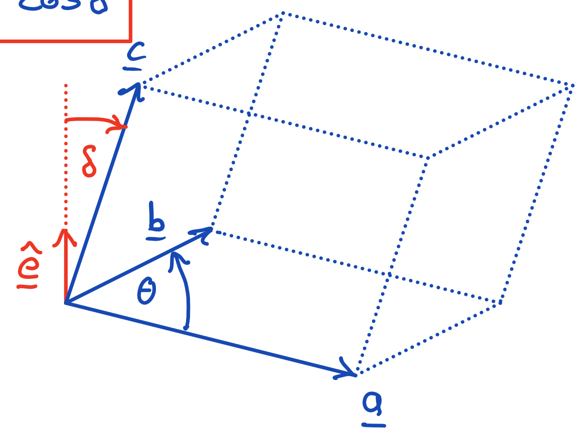
Triple scalar product $\quad \underline{a}, \underline{b}, \underline{c} \in V^3$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}||\underline{b}||\underline{c}| \sin\theta \cos\delta}$$

θ angle from $\underline{a}$ to $\underline{b}$

$\hat{\underline{e}}$ right handed normal to $\underline{a}$ and b

θ angle from $\hat{\underline{e}}$ to $\underline{c}$



$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{0} \Rightarrow \underline{a}, \underline{b}, \underline{c}$ linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form right handed system

$(\underline{a} \times \underline{b}) \cdot \underline{c} < 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form left handed system

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b}}$$

$\Rightarrow$ Volume of parallelepiped spanned by $\underline{a}, \underline{b}, \underline{c}$

Q: $(\underline{a} \times b) \cdot c \overset{?}{=} (\underline{b} \times \underline{a}) \cdot c$

Triple vector product

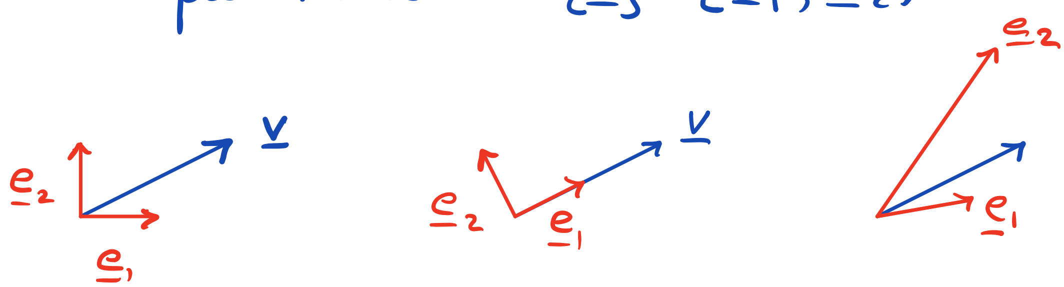
This may be new – we'll talk more about it later

$$\boxed{\begin{aligned} (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{a} \cdot \underline{c})\underline{b} - (b \cdot \underline{c})\underline{a} \\ \underline{a} \times (\underline{b} \times \underline{c}) &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \end{aligned}}$$

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}\}$ that span the space.

Examples in 2D: $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2)$



many choices $\Rightarrow$ not unique

In this class we will always choose a right-handed orthonormal basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

ortho-normal: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

right handed: $(\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

$\Rightarrow$ called Cartesian reference frame

Components of a vector in a basis

Project $\underline{v}$ onto basis vectors to get components.

$\cdot \underline{e}_1 \quad \cdot \underline{e}_2 \quad \cdot \underline{e}_3$

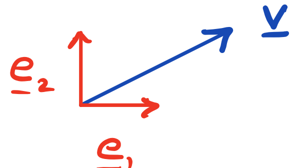
$$\boxed{\underline{v} = v_1 \underline{e}_1 + v_2 \cancel{\underline{e}_2} + v_3 \cancel{\underline{e}_3}}$$

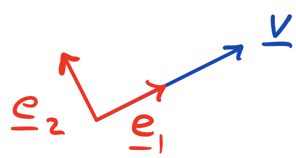
$$v_1 = \underline{v} \cdot \underline{e}_1 \qquad\qquad [\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
$$v_2 = \underline{v} \cdot \underline{e}_2$$
$$v_3 = \underline{v} \cdot \underline{e}_3$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

The distinction between a vector and its representation is important for this class.

Example:





$$|\underline{v}| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad\qquad [\underline{v}] = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad\qquad |\underline{v}| = \sqrt{\sqrt{5}^2 + 0^2} = \sqrt{5}$$

The vector is the same but representation is not.

Index Notation

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i \equiv a_i \underline{e}_i$$

Here the sum is always to 3!

1 Dummy index

is repeated twice in a term

$\Rightarrow$ Einstein summation convention

$$\sum_{i=1}^{N} a_i b_i \equiv a_i b_i$$

Note: symbol for index does not matter

$$\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k$$

$\Rightarrow$ rename dummy indices

2) Free index

A free index occurs only once

Example: $a_i = c_j b_j b_i$ i = free index

j = dummy index

free index represents a group of equations

$i=1: \quad a_1 = \sum_{j=1}^{3} c_j b_j \; b_1$

$i=2: \quad a_2 = \sum_{j=1}^{3} c_j b_j \; b_2$

$i=3: \quad a_3 = \sum_{j=1}^{3} c_j b_j \; b_3$

$\Big\}$ 3 equations

Basis: $\{\underline{e}_1 \; \underline{e}_2 \; \underline{e}_3\} = \{\underline{e}_i\}$

Note:
- all terms must have same free indices
- more than one free index (A_{ij})
- same symbol cannot be used for free and dummy index
- dummy indices can only be repeated twice

What is wrong?

1) $a_i = b_j$ — free indices are not same

2) $a_i b_j = c_i d_j d_j$ — j is used both as dummy & free index

3) $\underline{a_i b_j} = c_i \underline{c_k d_k} d_j + d_p c_\ell c_\ell d_q$

4) $a_i = b_k c_k d_k e_i$ k = is repeated 3 times

To express vector operations we need to introduce new symbols

Kronecker delta

For any frame $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

just consequence of orthonormal basis

Properties of Kron. delta:

$$\delta_{ij} = \delta_{ji} \qquad \text{symmetry}$$

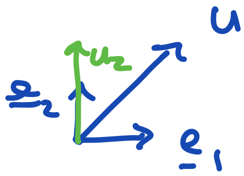
$$\underline{e}_i = \delta_{ij}\,\underline{e}_j \qquad \text{transfer property}$$

$$(\underline{e}_1 = \delta_{1j}\,\underline{e}_j)$$

Example: Project vector on basis vector

$$\underline{u} = u_i \underline{e}_i$$

$$\underline{u} \cdot \underline{e}_j = (u_i \underline{e}_i) \cdot \underline{e}_j = u_i \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} = u_i \delta_{ij} = u_j$$



$j = 2$

Example: scalar product

$\underline{a} \cdot \underline{b} = ?$ $\qquad \underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j$

$$(a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} = a_i b_j \delta_{ij}$$

$$= a_i b_i = a_j b_j$$

$$= \sum_{i=1}^{3} a_i b_i$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\Rightarrow$ Kronecker delta expresses scalar product in index notation

Permutation symbol (Levi-Civita)

To express cross product we introduce

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } \quad ijk \in \{123, 231, 312\} \ \underline{\text{even}} \text{ perm. } 123 \\ -1 & \text{if } \quad ijk \in \{321, 213, 132\} \ \underline{\text{odd}} \text{ perm. } 123 \\ 0 & \text{if repeated index} \end{cases}}$$

Flipping any two indices changes sign

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

Alternative definition

$$\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k$$

$$\epsilon_{ijk} = \det([\underline{e}_i, \underline{e}_j, \underline{e}_k])$$

For any orthonormal frame

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

Example: Vector product

$\underline{a} \times \underline{b} = \underline{c}$ $\quad \underline{a} = a_i \underline{e}_i \quad \underline{b} = b_j \underline{e}_j \quad \underline{c} = c_k \underline{e}_k$

what is c_k in terms of a_i b_j

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = a_i b_j \underbrace{\underline{e}_i \times \underline{e}_j}_{\epsilon_{ijk} \underline{e}_k}$$

$$= a_i b_j \epsilon_{ijk} \underline{e}_k = \epsilon_{ijk} a_i b_j \underline{e}_k = c_k \underline{e}_k$$

$$\Rightarrow \boxed{c_k = \epsilon_{ijk} a_i b_j}$$

k - free index

i & j dummies

$$c_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ij1} a_i b_j$$

$$= \cancel{\epsilon_{111}}^{\,0} a_1 b_1 + \cancel{\epsilon_{121}}^{\,0} a_1 b_2 + \dots$$