Lecture 10: Mechanical Equilibrium

Logistics: - no office hrs tomorrow

- HW4 ? due Friday end of day

Last time: Tensor calculus

Scalar gradient: $\nabla\phi = \phi_{,i}\,\underline{e}_i$ $\quad \frac{\partial\phi}{\partial x_i}$

Vector gradient: $\nabla\underline{v} = v_{i,j}\,\underline{e}_i \otimes \underline{e}_j$

Divergence of vector: $\nabla\cdot\underline{v} = \mathrm{tr}(\nabla\underline{v}) = v_{i,i}$

Divergence of tensor: $\nabla\cdot\underline{\underline{S}} = S_{ij,j}\,\underline{e}_i$

Product rules: $\nabla\cdot(\phi\,\underline{\underline{S}}) = \underline{\underline{S}}\,\nabla\phi + \phi\,\nabla\cdot\underline{\underline{S}}$

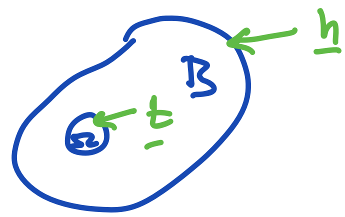
Curl: $\nabla\times\underline{v} = \boxed{\epsilon_{ijk}\,v_{i,k}\,\underline{e}_j}$

$\nabla\times\nabla\phi = 0 \qquad \nabla\cdot(\nabla\times\underline{v}) = 0$

Today: Mechanical equilibrium

Mechanical Equilibrium

Body at rest under the influence of a const body force $\underline{b}$ and an external traction, $\underline{h}$.



Necessary cond. for mech. eqbm

resultant force and torque must vanish for every subset Ω of the body B

$$\boxed{\begin{aligned} \underline{r}[\Omega] &= \underline{r}_b[\Omega] + \underline{r}_s[\partial\Omega] = \int_\Omega \rho \underline{b}\, dV + \oint_{\partial\Omega} \underline{t}\, dA = \underline{0} \\ \underline{\tau}[\Omega] &= \underline{\tau}_b[\Omega] + \underline{\tau}_s[\partial\Omega] = \int_\Omega \underline{x} \times \rho \underline{b}\, dV + \oint_{\partial\Omega} \underline{x} \times \underline{t}\, dA = 0 \end{aligned}}$$

show $\underline{r}[\Omega] = 0 \Rightarrow \underline{\tau}[\Omega]$ is independent of $\underline{z}$

$\Rightarrow$ set $\underline{z} = \underline{0}$

Local eqbm equations

$\underline{\underline{\sigma}}$ is continuously differentiable

ρ, $\underline{b}$ are continuous

$$\boxed{\begin{aligned} &\nabla \cdot \underline{\underline{\sigma}}(\underline{x}) + \rho(\underline{x})\, \underline{b}(\underline{x}) = 0 \\ &\underline{\underline{\sigma}}^T(\underline{x}) = \underline{\underline{\sigma}}(x) \end{aligned}}$$

for all $x \in B$

in components

$$\sigma_{ij,j} + \rho b_i = 0$$

$$\sigma_{ij} = \sigma_{ji}$$

To show this $\underline{t} = \underline{\underline{\sigma}}\, \underline{n}$

$$\underline{r}[\Omega] = \int_\Omega \rho \underline{b}\, dV + \oint_{\partial\Omega} \underline{\underline{\sigma}}\, \underline{n}\, dA$$

$\underline{n}$ = outward normal

using the tensor version of divergence Thm

$$\oint_{\partial\Omega} \underline{\underline{\sigma}}\, \underline{n}\, dA = \int_\Omega \nabla \cdot \underline{\underline{\sigma}}\, dV$$

$$\int_\Omega (\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b})\, dV = 0$$

because Ω is arbitrary $\Rightarrow$ integrand is zero

$$\Rightarrow \quad \nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b} = 0$$

substitute $\underline{t} = \underline{\underline{\sigma}}\, n$ into $\underline{I}[\Omega]$

$$\underline{I}[\Omega] = \int_{\partial\Omega} \underline{x} \times (\underline{\underline{\sigma}}\,\underline{n})\, dA + \int_{\Omega} \underline{x} \times \rho\underline{b}\, dA = 0$$

previous result: $\rho\underline{b} = -\nabla\cdot\underline{\underline{\sigma}}$

$$\int_{\partial\Omega} \underline{x} \times (\underline{\underline{\sigma}}\,\underline{n})\, dA + \int_{\Omega} \underline{x} \times (-\nabla\cdot\underline{\underline{\sigma}})\, dV$$

to simplify we define: $R_{il} = \epsilon_{ijk} x_j \sigma_{kl}$

$$\underline{\underline{R}}\,\underline{n} = \underline{x} \times (\underline{\underline{\sigma}}\,\underline{n})$$

$$\int_{\partial\Omega} \underline{\underline{R}}\,\underline{n}\, dA - \int_{\Omega} \underline{x} \times (\nabla\cdot\underline{\underline{\sigma}})\, dV = 0$$

apply tensor div. theorem

$$\int_{\Omega} \nabla\cdot\underline{\underline{R}} - \underline{x} \times (\nabla\cdot\underline{\underline{\sigma}})\, dV = 0$$

by arbitraryness of Ω

$$\nabla\cdot\underline{\underline{R}} - \underline{x} \times (\nabla\cdot\underline{\underline{\sigma}}) = 0$$

in index notation: $\nabla \cdot \underline{\underline{R}} = R_{il,l}$

$R_{il} = \epsilon_{ijk} x_j \sigma_{kl}$

$(\epsilon_{ijk} x_j \sigma_{kl})_{,l} - \underline{\epsilon_{ijk} x_j \sigma_{kl,l}} = 0$

differentiate first term

$(\epsilon_{ijk} x_j \sigma_{kl})_{,l} = \epsilon_{ijk} x_{j,l} \sigma_{kl} + \underline{\epsilon_{ijk} x_j \sigma_{kl,l}}$

$\epsilon_{ijk} x_{j,l} \sigma_{kl} = 0$ $\qquad x_{j,l} = \frac{\partial x_j}{\partial x_l} = \delta_{jl}$

$\epsilon_{ijk} \delta_{jl} \sigma_{kl} = 0$ $\qquad \frac{\partial x_1}{\partial x_1} = 1 \quad \frac{\partial x_2}{\partial x_3} = 0$

$\Rightarrow \boxed{\epsilon_{ijk} \sigma_{kj} = 0}$

If $\epsilon_{ijk} \sigma_{kj} = 0$ by prop. $\epsilon_{ijk} = -\epsilon_{ikj}$

$\epsilon_{ikj} \sigma_{jk} = 0$

$0 = \epsilon_{ijk} \sigma_{kj} + \epsilon_{ikj} \sigma_{jk} = \epsilon_{ijk}(\sigma_{kj} - \sigma_{jk}) = 0$

because we can always choose i to bee

d-stinct from j & k $\Rightarrow$ $\epsilon_{ijk} \neq 0$ $j \neq k$

$\Rightarrow$ $\sigma_{kj} = \sigma_{jk}$ $\Rightarrow$ $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$

resultant force $= 0$ $\Rightarrow$ $\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$

resultant torque $= 0$ $\Rightarrow$ $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$

Unknowns: $\underline{\underline{\sigma}}$ = 9 unknows σ_{ij}'s

$\underline{F}[\Omega] \Rightarrow$ 3 equations $\sigma_{ij,j} + \rho b_i = 0$ $i = 1, 2, 3$

$\underline{T}[\Omega] \Rightarrow$ $\left.\begin{aligned} \sigma_{13} &= \sigma_{31} \\ \sigma_{23} &= \sigma_{32} \\ \sigma_{12} &= \sigma_{21} \end{aligned}\right\}$ 3 equations

9 unknowns but only 6 equations

cannot be solved !

What is missing is a constitutive relation

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{u})$ $\underline{u}$ = displacements (elasticity)

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{v})$ $\quad$ $\underline{v}$ - velocities (fluids)

If $\underline{r}[\Omega] = \underline{0}$ then $\underline{T}[\Omega]$ independent of $\underline{z}$

$$\begin{aligned}
\underline{T}_{\underline{z}}[\Omega] &= \int_{\Omega} (\underline{x}-\underline{z}) \times \rho \underline{b}\, dV + \int_{\partial\Omega} (\underline{x}-\underline{z}) \times \underline{t}\, dA \\
&= \int_{\Omega} \underline{x} \times \rho \underline{b} - \underline{z} \times \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{x} \times \underline{t} - \underline{z} \times \underline{t}\, dA \\
&= \int_{\Omega} \underline{x} \times \rho b\, dV + \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA - \int_{\Omega} \underline{z} \times \rho b\, dV - \int_{\partial\Omega} \underline{z} \times \underline{t}\, dA
\end{aligned}$$

$$= \qquad - \underline{z} \times \underbrace{\left(\int_{\Omega} \rho \underline{b}\, dV + \int_{\partial\Omega} t\, dA \right)}_{\underline{r}[\Omega]}$$

$$\boxed{\underline{T}[\Omega] = \int_{\Omega} \underline{x} \times \rho b\, dV + \oint_{\partial\Omega} \underline{x} \times t\, dA} - \underline{z} \times \cancelto{0}{\underline{r}[\Omega]}$$