Lecture 11: Kinematics

Logistics: - HW4 due tomorrow

- HW5 posted today

Last time: - Equilibrium Eqns

$$\underline{r}[\Omega] = \underline{0} : \quad \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$$

$$\underline{\tau}[\Omega] = \underline{0} : \quad \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

- $\underline{r}[\Omega] = 0 \Rightarrow \underline{\tau}[\Omega]$ is independent of $\underline{z}$

Today: - New topic

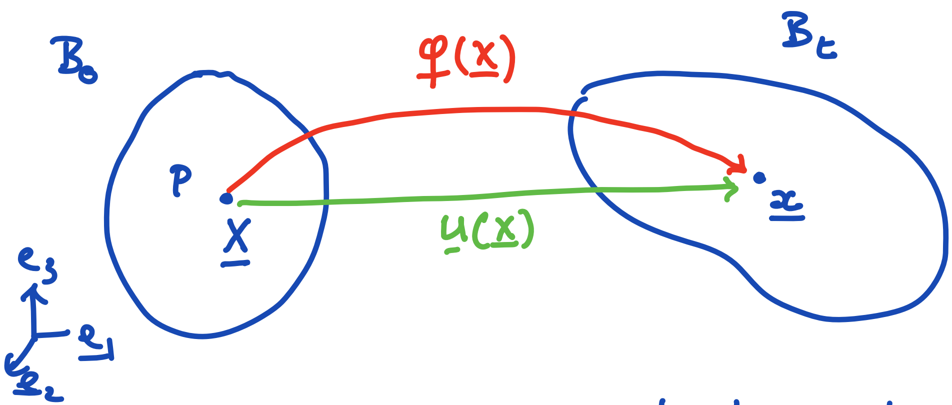
Kinematics - description of deformation

Kinematics

Study of the geometry of motion without the consideration of mass or stress.

$\Rightarrow$ Quantify strain and rate of strain

Deformation Mapping



B_0 = body reference, initial, material, undeformed configuration

P is material point (atom)

$\underline{X}$ = is location of P in B_0

B_t = body in current, spatial or deformed configuration

$\underline{x}$ = is location P in B_t

$\varphi(\underline{X})$ = deformation mapping

$\underline{u}(\underline{X})$ = displacement

$\{\underline{e}_i\}$ = ref. frame

Convention

Upper case quantities → ref. config $\underline{X}$

lower case quantities → current config $\underline{x}$

$\underline{X} = X_I \underline{e}_I$ $\qquad \underline{e}_I = \underline{e}_i$ use just one frame

$\underline{x} = x_i \underline{e}_i$

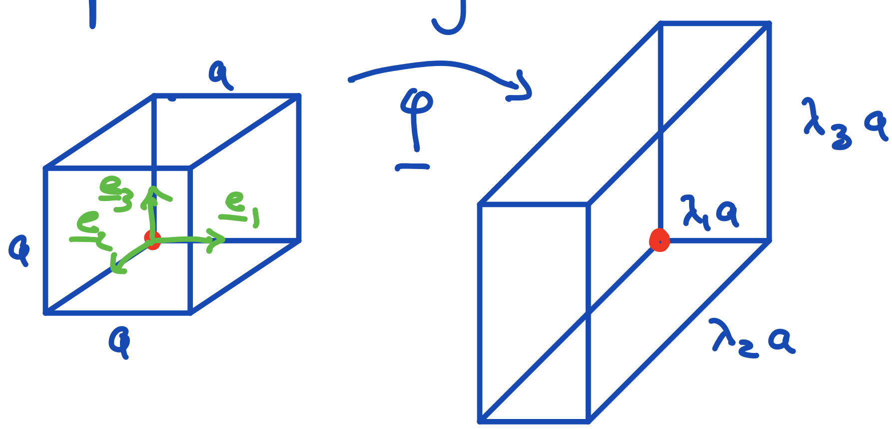
Definition of deformation mapping

$$\boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \varphi_i(\underline{X})\, \underline{e}_i}$$

Definition of displacement

$$\boxed{\underline{u}(\underline{X}) = \underline{\varphi}(\underline{X}) - \underline{X} = u_i(\underline{X})\, \underline{e}_i}$$

Example: Streching cube



deformation map: $x_1 = \lambda_1 X_1 + v_1$

$$x_2 = \lambda_2 X_2 + v_2$$

$$x_3 = \lambda_3 X_3 + v_3$$

λ_i = stretch ratios

$\underline{v}$ = translation (only important in presence of ~~for~~ body force)

$\underline{v} = \underline{0}$

$$\underline{x} = \varphi(\underline{X}) = \lambda_1 X_1 \underline{e}_1 + \lambda_2 X_2 \underline{e}_2 + \lambda_3 X_3 \underline{e}_3$$

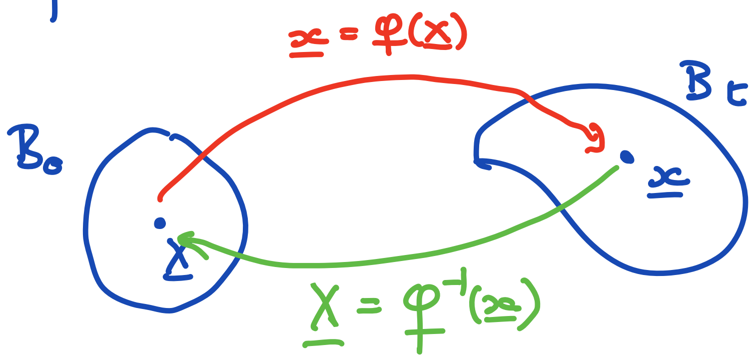
$$= \Lambda_{ij} X_j \underline{e}_i$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$$\underline{x} = \underline{\underline{\Lambda}}\, \underline{X}$$

Inverse Mapping

If φ is admissible $\Rightarrow$ well defined inverse φ^{-1}



Inverse def. map: $\boxed{\underline{X} = \varphi^{-1}(\underline{x}) = \varphi^{-1}_I(\underline{x})\,\underline{e}_I}$

Measures of strain

In 1D we have simple measures:

reference: $\underset{L}{\vdash\!\!-\!\!-\!\!\dashv}\overset{\Delta L}{\sim}$ $\qquad \Delta L = \ell - L$

deformed: $\underset{\ell}{\vdash\!\!-\!\!-\!\!-\!\!\dashv}$

engineering strain: $e = \frac{\Delta L}{L} = \frac{\ell - L}{L}$

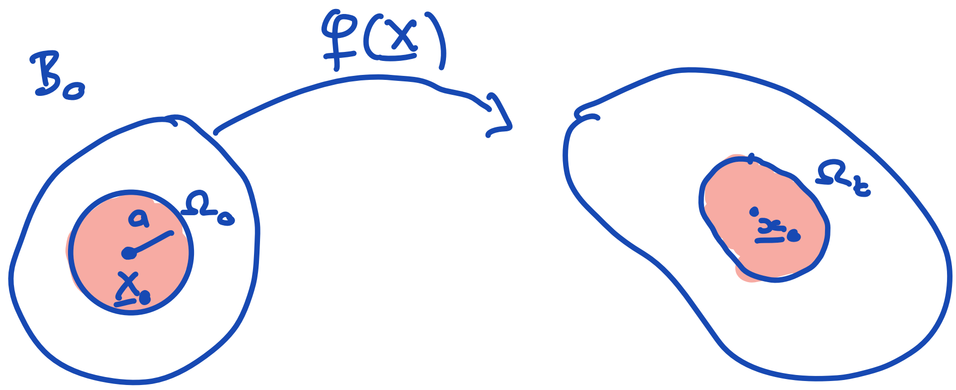
strech ration: $\lambda = \frac{\ell}{L} \qquad e = \lambda - 1$

Hencky strain: $\varepsilon = \ln(\lambda)$

...

Description of deformation is not unique!

Find general 3D approach that does not assume that deformation is small.



Deformation gradient

Natural way to quantify local strain

$$\boxed{\underline{\underline{F}}(\underline{X}) = \nabla\varphi(\underline{X})} \qquad F_{ij} = \frac{\partial \varphi_i}{\partial X_j}$$

Expand around $\underline{X}_0$

$$\varphi(\underline{X}) = \varphi(\underline{X}_0) + \nabla\varphi(\underline{X}_0)(\underline{X} - \underline{X}_0) + \text{h.o.t.}$$

$$= \underbrace{\varphi(\underline{X}_0) - \nabla\varphi(\underline{X}_0)\,\underline{X}_0}_{\underline{c}} + \underbrace{\nabla\varphi(\underline{X}_0)}_{\underline{\underline{F}}(\underline{X}_0)}\,\underline{X}$$

Locally we approximate φ as

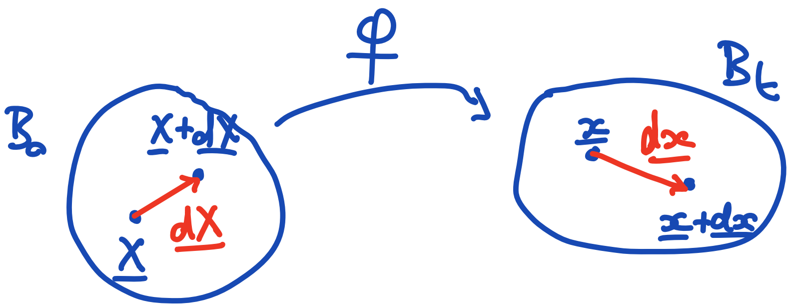
$$\boxed{\varphi(\underline{X}) \approx \underline{c} + \underline{\underline{F}}(\underline{X}_0)\,\underline{X}}$$

$\underline{\underline{F}}(\underline{X}_0)$ characterizes local deformation around $\underline{X}_0$

Homogeneous def. $\Rightarrow$ $\underline{\underline{F}}$ is constant

$$\boxed{\underline{x} = \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}}$$

Mapping of line segment



$|dX| \ll 1$

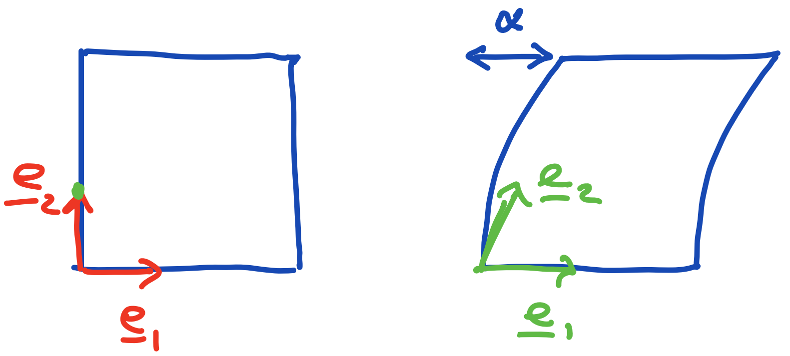
$|dx| \ll 1$

$$\cancel{\underline{x}} + d\underline{x} = \varphi(\underline{X}+d\underline{X}) \simeq \underbrace{\varphi(\underline{X})}_{\underline{x}} + \nabla\varphi(\underline{X})\, d\underline{X} = \cancel{\underline{x}} + \underbrace{\nabla\varphi(\underline{X})}_{\underline{\underline{F}}}\,\underline{X}$$

$$\boxed{\begin{aligned} d\underline{x} &= \underline{\underline{F}}(\underline{X})\, d\underline{X} \\ dx_i &= F_{iJ}\, dX_J \end{aligned}}$$

$\underline{\underline{F}}$ maps material vectors into spatial vectors.

Example : Shear deformation



shearing a deck of cards

$$\varphi(X) = [\underbrace{X_1 + \alpha X_2^2}_{x_1},\ \underbrace{X_2}_{x_2}]^T$$

$$\nabla\varphi = \underline{\underline{F}} = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix}$$

$$\underline{e}_1 : \quad \underline{\underline{F}}\,\underline{e}_1 = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{e}_2 : \quad \underline{\underline{F}}\,\underline{e}_2 = \begin{bmatrix} 2\alpha X_2 \\ 1 \end{bmatrix} \overset{X_2=1}{=} \begin{bmatrix} 2\alpha \\ 1 \end{bmatrix}$$

rotated and stretched

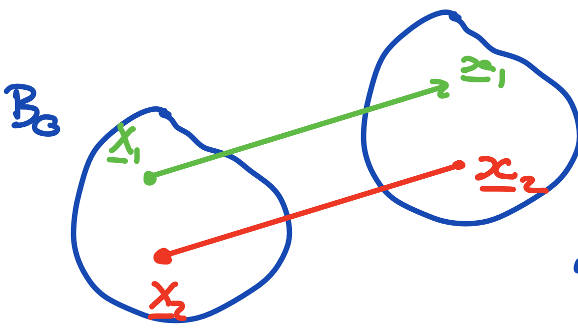
$e_2 = \underline{dX}$

$\underline{X}_0 = \underline{0}$

Translation

φ is a translation if $\underline{\underline{F}} = \underline{\underline{I}}$ so that

$$\underline{x} = \underline{c} + \underline{\underline{I}}\,\underline{X} = \underline{c} + \underline{X}$$



Each point in B_0 is shifted along $\underline{c}$

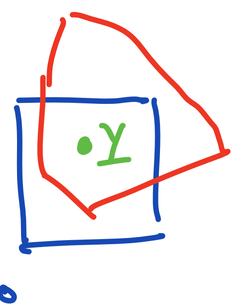
~~with~~

Fixed point

Deformation has a fixed point at $\underline{Y}$ if

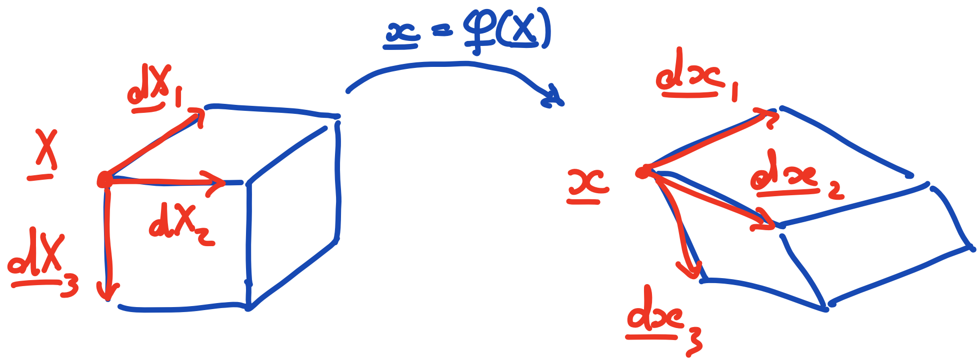
$$\boxed{\varphi(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X}-\underline{Y})}$$

so that $y = \varphi(\underline{Y}) = \underline{Y}$



Note: Fixed point $\underline{Y}$ ~~must~~ need not be in $\underline{B}_0$

Volume changes.



Volumes: $dV_X = (\underline{dX}_1 \times \underline{dX}_2) \cdot \underline{dX}_3$

$$dV_x = (\underline{dx}_1 \times \underline{dx}_2) \cdot \underline{dx}_3$$

$$= \det([\underline{dx}_1\ \underline{dx}_2\ \underline{dx}_3])$$

substitute: $\underline{dx}_i = \underline{\underline{F}}\, \underline{dX}_I$

$$dV_x = \det([\underline{\underline{F}}\,\underline{dX}_1, \underline{\underline{F}}\,\underline{dX}_2, \underline{\underline{F}}\,\underline{dX}_3]) \qquad \underline{\underline{dX}} = [\underline{dX}_1, \underline{dX}_2 \underline{dX}_3]$$

$$\det(\underline{\underline{F}}\,\underline{\underline{dX}}) = \det(\underline{\underline{F}}) \det(\underline{\underline{dX}})$$

$$= \det(\underline{\underline{F}}) \underbrace{(\underline{dX}_1 \times \underline{dX}_2) \cdot \underline{dX}_3}_{dV_X}$$

$$\boxed{dV_x = \det(\underline{\underline{F}})\, dV_X}$$

The field $J(\underline{X}) = \det(\underline{\underline{F}}(\underline{X})) = \dfrac{dV_x}{dV_X}$ is called the Jacobian of φ and measures the volume strain

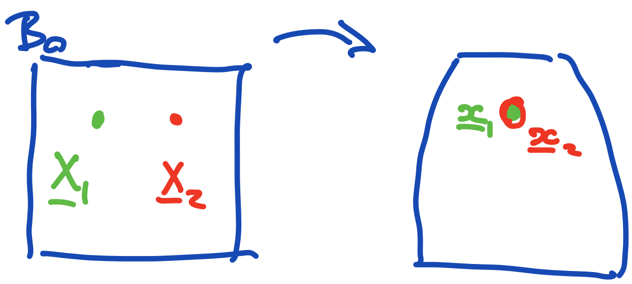
$J(\underline{X}) > 1$: volume increases $dV_x > dV_X$

$J(\underline{X}) < 1$: volume decreases

$J(\underline{X}) = 1$: no volume change

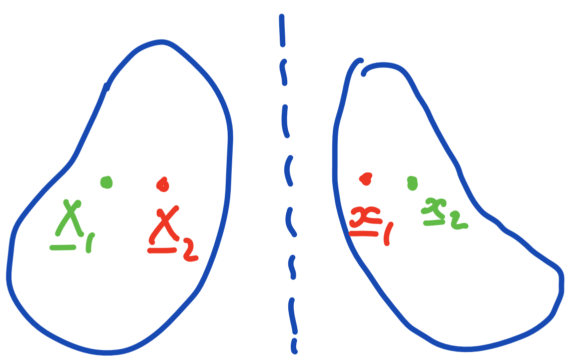
Admissible deformation

1) $\varphi : B_0 \to B_t$ is one to one and onto



two separate points in B_0 cannot be mapped to same point in B_t

2) $\det(\nabla\varphi) > 0$



body cannot be deformed into its mirror image