Lecture 12: Analysis of local deformation

Logistics: - HW5 is due Th

Last time: - Introduction to kinematics & Strain

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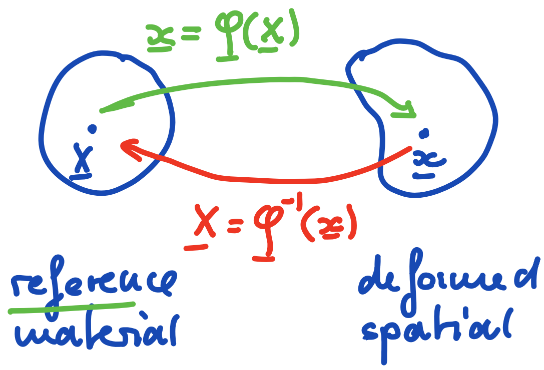
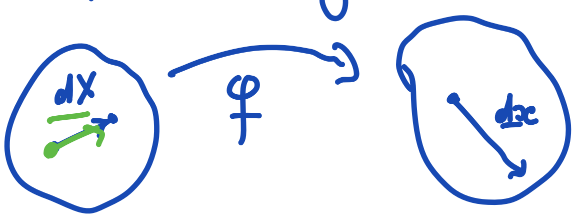


deformation map

- Quantify strain

$\Rightarrow$ deformation gradient $\underline{F} = \nabla \underline{\varphi}$



$$\boxed{d\underline{x} = \underline{F}\, d\underline{X}}$$

- Jacobian: $J(\underline{X}) = \det(\underline{F}(\underline{X})) = \dfrac{dV_x}{dV_X}$

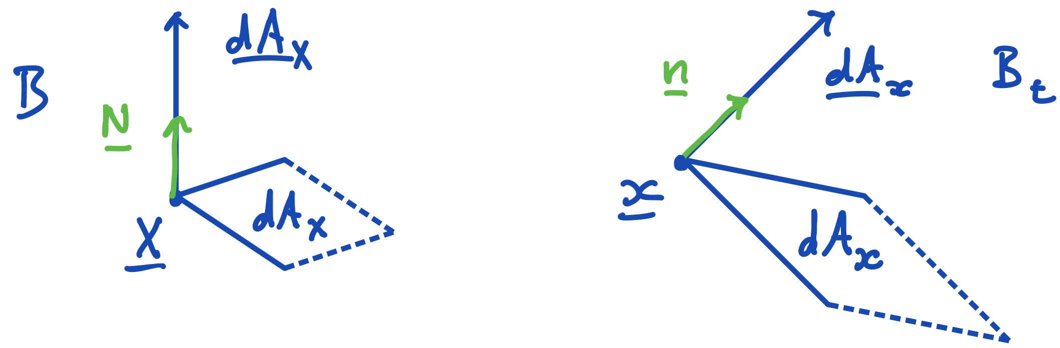
Today: - Area changes

- Polar decomposition, Tensor square root

- Decompose $\underline{F}$ to find strain tensor

Surface area changes

How do surfaces change during deformation



surface normals: $|\underline{N}| = |\underline{n}| = 1$

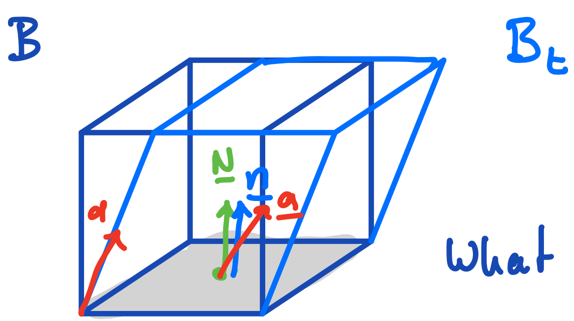
surface vector elements:

$$\underline{dA}_X = \underline{N}\, dA_X$$

$$\underline{dA}_x = \underline{n}\, dA_x$$

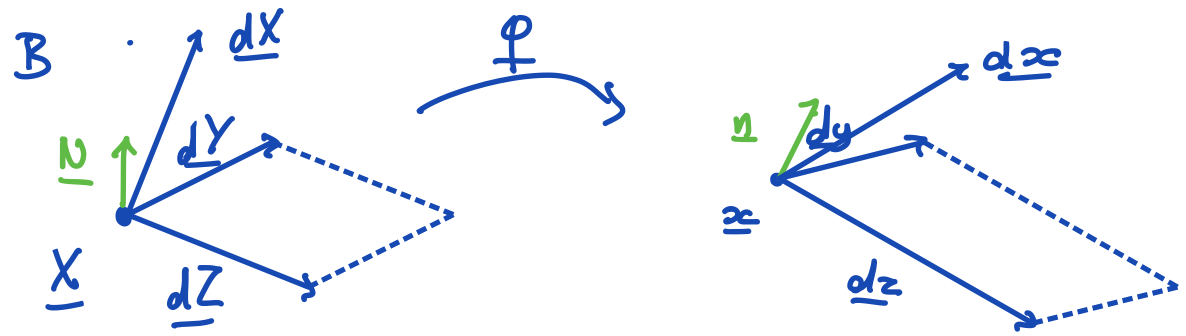
Important: $\underline{n} \neq \underline{\underline{F}}\, \underline{N}$

Example of simple shear



$$a = \underline{\underline{F}}\, \underline{N}$$

What is the relation between $\underline{N}$ and $\underline{n}$?



$$\underline{N} \cdot d\underline{X} \neq 0$$

$$d\underline{A}_X = d\underline{Y} \times d\underline{Z} \qquad d\underline{A}_x = d\underline{y} \times d\underline{z}$$

$$dV_X = d\underline{A}_X \cdot d\underline{X} \qquad dV_x = d\underline{A}_x \cdot d\underline{x}$$

Change in volume: $dV_x = J\, dV_X$

$$d\underline{A}_x \cdot d\underline{x} = J\,(d\underline{A}_X \cdot d\underline{X}) \qquad d\underline{x} = \underline{\underline{F}}\, d\underline{X}$$

$$d\underline{A}_x \cdot (\underline{\underline{F}}\, d\underline{X}) - J\, d\underline{A}_X \cdot d\underline{X} = 0$$

$$\underline{\underline{F}}^T d\underline{A}_x \cdot d\underline{X} - J\, d\underline{A}_X \cdot d\underline{X} = 0$$

$$(\underline{\underline{F}}^T d\underline{A}_x - J\, d\underline{A}_X) \cdot d\underline{X} = 0 \qquad d\underline{X} \text{ is arbitrary}$$

$$F^T d\underline{A}_x = J\, d\underline{A}_X$$

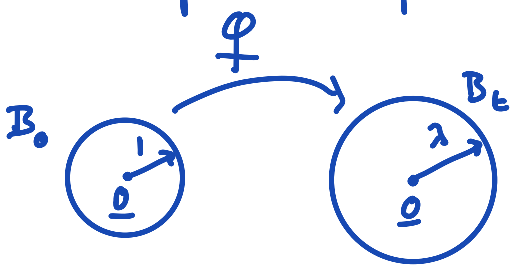
$$\boxed{\begin{aligned} d\underline{A}_x &= J\, \underline{\underline{F}}^{-T} d\underline{A}_X \\ \underline{n}\, dA_x &= J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X \end{aligned}}$$

Nanson's formula

$$n = \frac{J\, dA_X}{dA_x}\, F^{-T} N$$

$$- \underbrace{dA_x}_{\text{scaling}} \underbrace{= -}_{\text{direction}}$$

Example: Expanding sphere $\quad V = \frac{4}{3}\pi R^3$



$$V_0 = \frac{4\pi}{3}$$

$$V_t = \frac{4\pi}{3}\lambda^3$$

Deformation map: $\quad \underline{x} = \underline{\varphi}(\underline{X}) = \lambda \underline{X} \qquad \lambda > 0$

$$\underline{\underline{F}} = \nabla \underline{\varphi} = \lambda \underline{\underline{I}} \qquad \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial(\lambda X_i)}{\partial X_j}$$

$$J = \det(\underline{\underline{F}}) = \det(\lambda \underline{\underline{I}})$$

$$= \lambda^3 \det(\underline{\underline{I}}) = \lambda^3$$

$$V_t = J\, V_0 = \frac{4}{3}\pi \lambda^3 \; \checkmark$$

Change in area: $\quad A_0 = 4\pi \qquad A_t = 4\pi\lambda^2$

$$A_t / A_0 = \lambda^2$$

$$\underline{\underline{F}}^{-T} = (\lambda \underline{\underline{I}})^{-T} = (\lambda \underline{\underline{I}})^{-1} = \frac{1}{\lambda}\underline{\underline{I}}$$

$$\underline{\underline{F}}\,\underline{\underline{F}}^{T} = \lambda \underline{\underline{I}}\, \frac{1}{\lambda}\underline{\underline{I}} = \underline{\underline{I}}$$

Nanson: $\quad \underline{n}\, dA_x = \underbrace{J\, \underline{\underline{F}}^{-T}}\, \underline{N}\, dA_X$

$$\lambda^3 \frac{1}{\lambda}\underline{\underline{I}}\,\underline{N}\, dA_X = \lambda^2 \underline{N}\, dA_X$$

$$\underline{n}\, dA_x = \lambda^2\, \underline{N}\, dA_X$$

$$\underline{n}\, \frac{dA_x}{dA_X} = \lambda^2\, \underline{N}$$

taking absolute value: $\quad \dfrac{dA_x}{dA_X} = \lambda^2$

$\rightarrow \quad \underline{n} = \underline{N}$

Polar decomposition

Any tensor $\underline{\underline{F}} \in \mathcal{V}^2$ with $\det(\underline{\underline{F}}) > 0$

has a right & left polar decomposition

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$$

where $\underline{\underline{R}}$ is a rotation

$$\left.\begin{aligned} \underline{\underline{U}} &= \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}} \\ \underline{\underline{V}} &= \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} \end{aligned}\right\} \quad \text{sym. pos. def.}$$

$\Rightarrow$ can diagonalize

if $\det(\underline{\underline{F}}) > 0 \qquad \underline{\underline{F}}\,\underline{v} \neq 0 \quad$ for $\underline{v} \neq 0$

$\det(\underline{\underline{F}}^T) > 0 \qquad \underline{\underline{F}}^T \underline{v} \neq 0 \quad$ for $\underline{v} \neq 0$

Requirement of admissible deformation ($\det \underline{\underline{F}} \neq 0$)

$\Rightarrow$ $\underline{\underline{U}}$, $\underline{\underline{V}}$ are s.p.d.

$$(\underline{\underline{F}}\,\underline{v}) \cdot (\underline{\underline{F}}\,\underline{v}) > 0$$

transpose

$$\underline{v} \cdot \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\text{sp.d}}\, \underline{v} > 0$$

$$(\underline{\underline{F}}^T \underline{v}) \cdot (\underline{\underline{F}}^T \underline{v}) > 0$$

$$\underline{v} \cdot \underbrace{\underline{\underline{F}}\,\underline{\underline{F}}^T}_{\text{spd}}\, \underline{v} > 0$$

Tensor square root:

if $\underline{\underline{C}}$ is s.p.d tensor with eigen pairs $(\lambda, \underline{v})$ there is a unique tensor $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$

$$\boxed{\underline{\underline{U}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\; \underline{v}_i \otimes \underline{v}_i}$$

$$\underline{\underline{A}}^3 = \underline{\underline{V}}^T \underline{\underline{\Lambda}}\, \overset{I}{\cancel{\underline{\underline{V}}\, \underline{\underline{V}}^T}}\, \underline{\underline{\Lambda}}\, \overset{I}{\cancel{\underline{\underline{V}}\, \underline{\underline{V}}^T}}\, \underline{\underline{\Lambda}}\, \underline{\underline{V}} = \underline{\underline{V}}^T \underline{\underline{\Lambda}}^3 \underline{\underline{V}}$$

$$\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

similar to "matrix" exponentiation.

$\Rightarrow$ $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

Show that $\underline{\underline{R}}$ is a rotation

1) $\det(\underline{\underline{R}}) = 1$

2) $\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}}$

Show $\underline{\underline{R}}$ is orthonormal $\qquad \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1}$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$$

$$\underline{\underline{R}}^T \underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T (\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-T} \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$$

$$\underline{\underline{U}}^{-1} \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\underline{\underline{U}}^2} \underline{\underline{U}}^{-1} = \underline{\underline{I}}$$

Show $\det(\underline{\underline{R}}) > 0$

$$\det(\underline{\underline{R}}) = \det(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \frac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$$

> 0

$\det(U) = \lambda_1 \lambda_2 \lambda_3$

$\Rightarrow$ $\underline{\underline{R}}$ is a rotation

Strech-rotation decomposition

Let φ be hom. def. with fixed point $\underline{y}$

so that $\quad \varphi(\underline{x}) = \underline{y} + \underline{\underline{F}}(\underline{x}-\underline{y})$

then we have

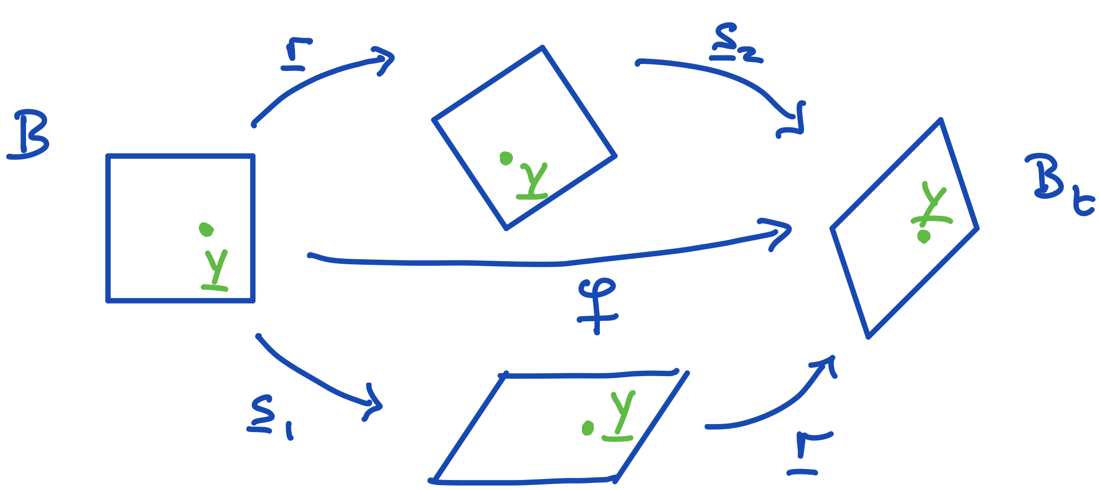
$$\boxed{\varphi = \underline{r} \circ \underline{s}_1 = \underline{s}_2 \circ \underline{r}}$$

$f(x)$ and $g(x)$ $\quad f \circ g = f(g(x))$

where $\quad \underline{r} = \underline{y} + \underline{\underline{R}}(\underline{x}-\underline{y})$ $\quad$ rotation around $\underline{y}$

$$\left.\begin{aligned} \underline{s}_1 &= \underline{y} + \underline{\underline{U}}(\underline{x}-\underline{y}) \\ \underline{s}_2 &= \underline{y} + \underline{\underline{V}}(\underline{x}-\underline{y}) \end{aligned}\right\} \text{ streches}$$

when $\quad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$ $\qquad \underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ $\quad \underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$



To see this consider

$$(\underline{r} \circ \underline{s}_1)(\underline{x}) = \underline{r}(\underline{s}_1(\underline{x})) = \underline{y} + \underline{\underline{R}}(\underline{s}_1(\underline{x}) - \underline{y})$$

$$= \underline{y} + \underline{\underline{R}}(\underbrace{\cancel{\underline{y}} + \underline{\underline{U}}(\underline{x} - \underline{y})}_{\underline{s}_1} - \cancel{\underline{y}})$$

$$= \underline{y} + \underbrace{\underline{\underline{R}}\,\underline{\underline{U}}}_{\underline{\underline{F}}}(\underline{x} - \underline{y})$$

$$\underline{\varphi}(\underline{x}) = \underline{y} + \underline{\underline{F}}(\underline{x} - \underline{y})$$

Streck tensors

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i \qquad \underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$

where $(\lambda, \underline{u})$ and (λ, v) are eigenpairs of $\underline{\underline{U}}$ & $\underline{\underline{V}}$

$\Rightarrow$ $\underline{\underline{U}}$ & $\underline{\underline{V}}$ have same eigenvalues but different eigen vectors

$$\Rightarrow \qquad \underline{v}_i = \underline{\underline{R}}\, \underline{u}_i$$

This decomposition allows us to extract only stretches that cause deformation from $\underline{\underline{F}}$

$$\underline{\underline{C}} = \underline{\underline{U}}^2$$ next time strain tensor