Lecture 13: Cauchy-Green Strain Tensor

Logistics: - HW 5 due

- HW 4 had issue with fault image

- HW 6 will be posted

Last time: Analysis of local deformation

- Changes in surface area: $\boxed{\underline{n}\, dA_x = J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X}$

- Polar Decomposition: $\underline{\underline{F}} = \underbrace{\underline{\underline{R}}\,\underline{\underline{U}}}_{\text{right}} = \underbrace{\underline{\underline{V}}\,\underline{\underline{R}}}_{\text{left}}$ $\det(F) > 0$

$\underline{\underline{R}}$ = rotation $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ streches

- Strech-rotation decomp.: $\underline{\varphi} \overset{\underline{\underline{F}}}{=} \underline{r} \circ \underline{s}_1 = \underline{s}_2 \circ \underline{r}$

Today: - right Cauchy-Green strain tensor $\underline{\underline{C}}$

- Interpretation of $\underline{\underline{C}}$

- Cauchy-Green strain relations

- Components of $\underline{\underline{C}}$

Cauchy-Green Strain Tensor

$\varphi : B \to B_t$ $\quad \underline{x} = \varphi(\underline{X}) = c + \underline{\underline{F}}\,\underline{X}$ $\qquad \underline{\underline{F}} = \nabla \varphi$

(right) Cauchy-Green strain tensor

$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2}$$

only contains information about streches

Can get $\underline{\underline{C}}$ from $\underline{\underline{F}}$ but not the other way

Why not use $\underline{\underline{U}}$? $\underline{\underline{C}}$ is much simpler to work with

Example:

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To get $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$ $\Rightarrow$ need to solve eigen value problem:

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 5-\mu & 4 \\ 0 & 4 & 5-\mu \end{vmatrix} = (1-\mu)(5-\mu)^2 - 16(1-\mu) = 0$$

$$\mu_{1/2} = 1 \qquad \mu_3 = 9$$

eigen vectors: $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\underline{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad \underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sum_{i=1}^{3} \sqrt{\mu_i}\, \underline{u}_i \otimes \underline{u}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Note: $\underline{\underline{F}} = \underline{\underline{U}}$ because there is no rotation.

Here μ are eigen values of $\underline{\underline{C}}$

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i$$

λ_i's principal streches

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \sum_{i=1}^{3} \lambda_i^2 \, \underline{u}_i \otimes \underline{u}_i$$

$\mu_i = \lambda_i^2$ eigen values of $\underline{\underline{C}}$ are squares of principal streches

$$x_i \underline{e}_i = F_{iJ} X_J \underline{e}_i \qquad \underline{x} = \underline{\underline{F}}\, \underline{X}$$

$C_{KL} = F_{iK} F_{iL}$ "material strain tensor"

Other strain tensors

I) $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ Green - Lagrange strain tensor

$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$ material tensor

$\Rightarrow$ linear theory

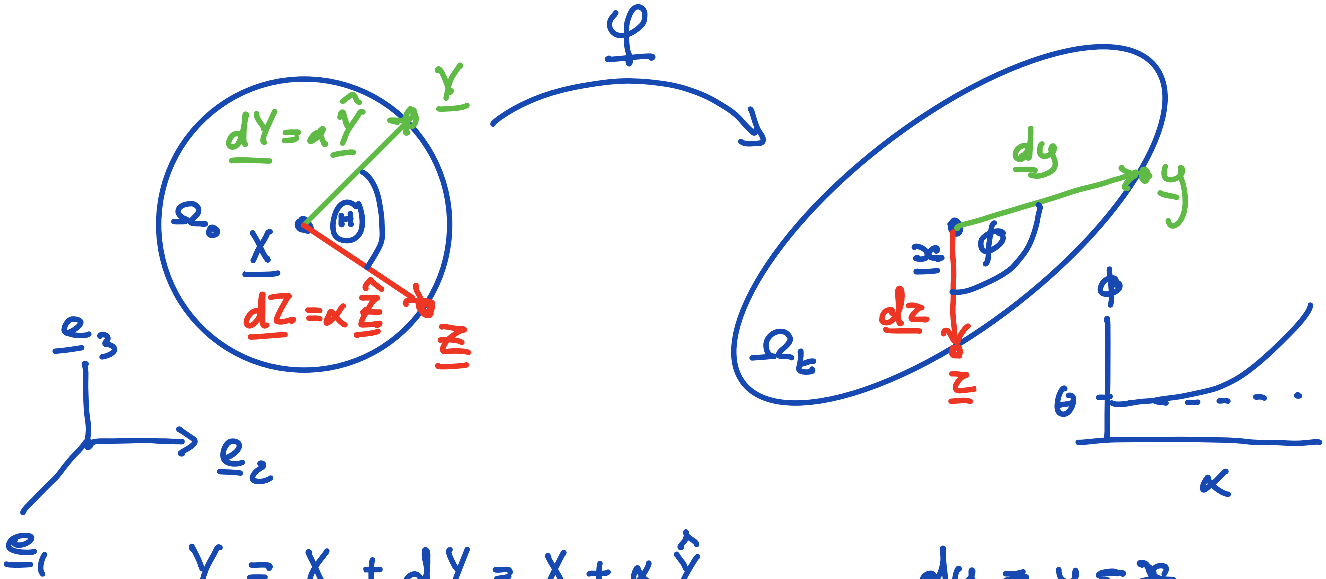
II) $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{V}^2$ left Cauchy Green strain ten.

$b_{kl} = F_{kI} F_{lI}$ "spatial tensor"

III) $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T} \underline{\underline{F}}^{-1})$ Euler - Almansi tensor

$e_{kl} = \frac{1}{2}(\delta_{kl} - F^{-1}_{Ik} F^{-1}_{Il})$ "spatial tensor"

Interpretation of $\underline{\underline{C}}$



$$\underline{Y} = \underline{X} + d\underline{Y} = \underline{X} + \alpha \underline{\hat{Y}} \qquad d\underline{y} = \underline{y} - \underline{x}$$

$$\underline{Z} = \underline{X} + d\underline{Z} = \underline{X} + \alpha \underline{\hat{Z}} \qquad d\underline{z} = \underline{z} - \underline{x}$$

In limit $\alpha \to 0$ $\quad \cos\phi \to \cos\theta(\underline{\hat{Y}}, \underline{\hat{Z}})$

Cauchy - Green Strain Relations

$$\boxed{\lambda(\underline{\hat{Y}}) = \sqrt{\underline{\hat{Y}} \cdot \underline{\underline{C}}\, \underline{\hat{Y}}}} \qquad \boxed{\cos\theta(\underline{\hat{Y}}, \underline{\hat{Z}}) = \frac{\underline{\hat{Y}} \cdot \underline{\underline{C}}\, \underline{\hat{Z}}}{\underbrace{\sqrt{\underline{\hat{Y}} \cdot \underline{\underline{C}}\, \underline{\hat{Y}}}}_{\lambda(\underline{\hat{Y}})}\ \underbrace{\sqrt{\underline{\hat{Z}} \cdot \underline{\underline{C}}\, \underline{\hat{Z}}}}_{\lambda(\underline{\hat{Z}})}}}$$

$\Rightarrow$ stretch at $\underline{X}$ in direction $\underline{\hat{Y}}$

I. Stretches

In the limit $\alpha \to 0$

$$\frac{|d\underline{y}|}{|d\underline{Y}|} \to \lambda(\hat{\underline{Y}}) \quad \text{and} \quad \frac{|d\underline{z}|}{|d\underline{Z}|} \to \lambda(\hat{\underline{Z}})$$

$$|d\underline{y}|^2 = d\underline{y} \cdot d\underline{y} = (\underline{\underline{F}}\, d\underline{Y}) \cdot (\underline{\underline{F}}\, d\underline{Y}) = d\underline{Y} \cdot \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\underline{\underline{C}}}\, d\underline{Y}$$

$$= d\underline{Y} \cdot \underline{\underline{C}}\, d\underline{Y} \qquad d\underline{Y} = \alpha \hat{\underline{Y}}$$

$$= \alpha^2 \; \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}$$

$$|d\underline{Y}|^2 = |\alpha \hat{\underline{Y}}|^2 = \alpha^2$$

$$\frac{|d\underline{y}|}{|d\underline{Y}|} = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}} = \lambda(\hat{\underline{Y}})$$

If $(\lambda_i, \hat{\underline{U}}_i)$ are eigenpair of $\underline{\underline{U}}$

$$(\underline{\underline{C}} - \lambda_i^2 \underline{\underline{I}})\, \hat{\underline{U}}_i = \underline{0} \qquad \text{eigen problem for } \underline{\underline{C}}$$

$$\underline{\underline{C}}\, \hat{\underline{U}}_i - \lambda_i^2 \hat{\underline{U}}_i = \underline{0}$$

$$\hat{\underline{U}}_i \cdot \underline{\underline{C}}\, \hat{\underline{U}}_i - \lambda_i^2 \underbrace{\hat{\underline{U}}_i \cdot \hat{\underline{U}}_i}_{1} = 0 \qquad \lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}$$

$$\lambda_i = \sqrt{\hat{\underline{U}}_i \cdot \underline{\underline{C}}\, \hat{\underline{U}}_i} \qquad \lambda(\hat{\underline{U}}_i) \qquad \hat{\underline{Y}} = \hat{\underline{U}}_i$$

$\Rightarrow$ proves that λ_i's are principal streches

Use argument similar to max/min normal stresses to show that λ_i's correspond to the max/min streches.

II) Shear

The shear $\gamma(\hat{\underline{Y}}, \hat{\underline{Z}})$ at $\underline{X}$ is the change in angle between the two directions $\hat{\underline{Y}}$ & $\hat{\underline{Z}}$

$$\boxed{\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})}$$

where $$\lim_{\alpha \to 0} \cos\phi = \cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}})$$

$$\underline{dy} \cdot \underline{dz} = |\underline{dy}||\underline{dz}| \cos\phi$$

$$\cos\phi = \frac{\underline{dy} \cdot \underline{dz}}{|\underline{dy}||\underline{dz}|}$$

$$d\underline{y} \cdot d\underline{z} = (\underline{\underline{F}}\, d\underline{Y}) \cdot (\underline{\underline{F}}\, d\underline{Z}) = d\underline{Y} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\, d\underline{Z}$$

$$= d\underline{Y} \cdot \underline{\underline{C}}\, d\underline{Z} = \alpha^2\, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}$$

$$|d\underline{y}| = \alpha\, \lambda(\hat{\underline{Y}}) = \alpha \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}} \qquad |d\underline{z}| = \alpha \sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}$$

$$\cos\phi = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}\ \sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}} \longrightarrow \cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}})$$

Components of $\underline{\underline{C}}$

For frame $\{\underline{e}_I\}$

$$\boxed{\begin{aligned} C_{II} &= \lambda^2(\underline{e}_I) \\ C_{IJ} &= \lambda(\underline{e}_I)\, \lambda(\underline{e}_J)\, \sin\gamma(\underline{e}_I, \underline{e}_J) \end{aligned}}$$

diagonal comp. are squared stretches in coordinate directions

off-diagonal components are related to shear between associated coor. dir.

Diagonal components:

$$\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}$$

$$\underline{\underline{A}} \quad \{\underline{e}_i\} \qquad A_{ij} = \underline{e}_i \cdot \underline{\underline{A}}\, \underline{e}_j$$

$$\Rightarrow \quad C_{II} = \underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_I = \lambda^2(\underline{e}_I)$$

Off diagonal components:

C_{IJ}

$$\cos\theta(\underline{e}_I, \underline{e}_J) = \frac{\underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_J}{\lambda(\underline{e}_I)\, \lambda(\underline{e}_J)} \qquad C_{IJ} = \underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_J$$

$$C_{IJ} = \lambda(\underline{e}_I)\, \lambda(\underline{e}_J) \cos\theta(\underline{e}_I, \underline{e}_J)$$

shear between two basis vectors

$$\gamma(\underline{e}_I, \underline{e}_J) = \underbrace{\Theta(\underline{e}_I, \underline{e}_J)}_{\frac{\pi}{2}} - \theta(\underline{e}_I, \underline{e}_J)$$

$$\theta(\underline{e}_I; \underline{e}_J) = \frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)$$

$$C_{IJ} = \lambda(\underline{e}_I)\, \lambda(\underline{e}_J) \underbrace{\cos\left(\frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)\right)}_{\sin(\gamma(\underline{e}_I, \underline{e}_J))}$$

$\Rightarrow$ Components of $\underline{\underline{C}}$ directly quantify strech and shear unlike components of $\underline{\underline{E}}$.