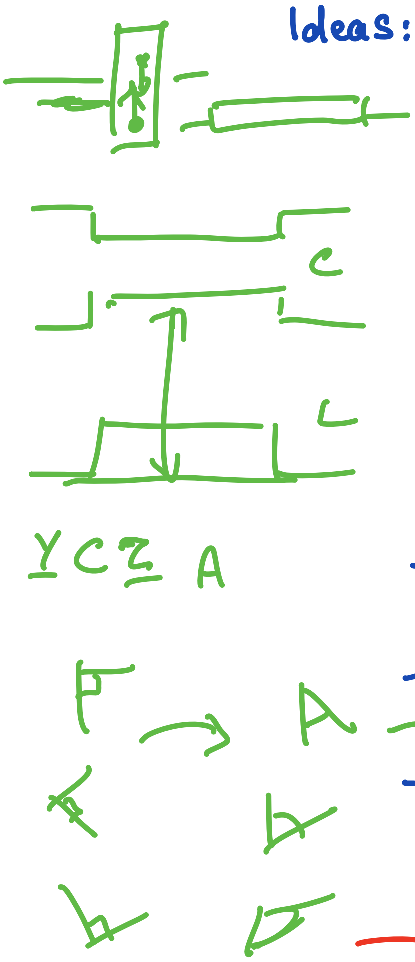
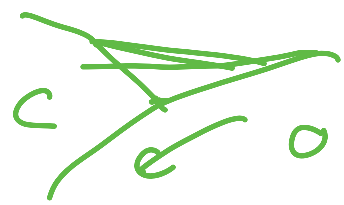
Lecture 14: Infinitesimal strain tensor

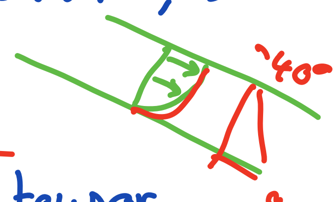
Logistics: - Need to think about little projects

Ideas:
- Compute normal to arbitrary fault
- Isostacy (Depth of ocean, height of mountains)
- Stresses on real faults (San Andreas, cascadia)
- Moment of inertia tensors
- Geologic strain markers (fossils, ooids)
- Critical taper theory
- tipping ice bergs
- Glacier sliding: T-variation, shear heating

... (your idea)







Last time: Cauchy-Green Strain tensor

$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2$ others $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$

Cauchy-Green strain relations

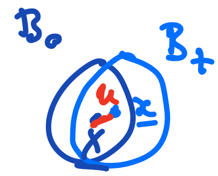
$\gamma = \Theta - \theta$

$$\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}} \qquad \cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}{\lambda(\hat{\underline{Y}})\,\lambda(\hat{\underline{Z}})}$$

$$C_{II} = \lambda^2(\underline{e}_I) \qquad C_{IJ} = \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\,\sin(\gamma(\underline{e}_I, \underline{e}_J))$$

Infinitesimal Strain Tensor

displacement: $\underline{u} = \underline{\varphi}(\underline{X}) - \underline{X}$



measure of strain

$$\boxed{\underline{\underline{\varepsilon}} = \text{sym}(\nabla \underline{u}) = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)}$$

infinitesimal strain tensor

Relate $\nabla \underline{u} = \underline{\underline{H}}$ to $\underline{\underline{F}} = \nabla \underline{\varphi}$ and $\underline{\underline{C}}$

$$\nabla \underline{u} = \nabla(\underline{\varphi} - \underline{X}) = \nabla \underline{\varphi} - \nabla \underline{X} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{F}} - \underline{\underline{I}} + (\underline{\underline{F}} - \underline{\underline{I}})^T\right) = \frac{1}{2}\left(\underline{\underline{F}} + \underline{\underline{F}}^T\right) - \underline{\underline{I}}$$

addition linear

$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ multiplication -> non-linear

$$\underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}}$$

$$\underline{\underline{C}} = (\nabla \underline{u} + \underline{\underline{I}})^T (\nabla \underline{u} + \underline{\underline{I}}) = (\nabla \underline{u}^T + \underline{\underline{I}})(\nabla \underline{u} + \underline{\underline{I}})$$

$$= \nabla \underline{u}^T \nabla \underline{u} + \underbrace{\nabla \underline{u}^T + \nabla \underline{u}}_{2\underline{\underline{\varepsilon}}} + \underline{\underline{I}}$$

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) - \frac{1}{2}\nabla \underline{u}^T \nabla \underline{u} = \underline{\underline{E}} - \frac{1}{2}\nabla \underline{u}^T \nabla \underline{u}}$$

usefull if $|\nabla \underline{u}| = O(\epsilon)$ $\quad 0 < \epsilon \ll 1$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) + O(\epsilon^2)$$

$\Rightarrow \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) \longrightarrow \underline{\underline{\varepsilon}}$ if $|\nabla \underline{u}| = O(\epsilon)$

$\underline{\underline{C}}$ nonlinear in $\nabla \underline{u}$ but $\underline{\underline{\varepsilon}}$ is linear $|\nabla \underline{u}|$

$\Rightarrow$ commonly used to describe strain in elastic solids

Interpretation of components of $\underline{\underline{\varepsilon}}$

$$\boxed{\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1 \quad (\text{no sum}) \qquad \varepsilon_{ij} \approx \frac{1}{2}\sin\gamma(\underline{e}_i, \underline{e}_j)} \qquad i \neq j$$

$\underline{\underline{C}} = 2\,\underline{\underline{\varepsilon}} + \underline{\underline{I}}$

Diagonal components:

$C_{ii} = 1 + 2\,\varepsilon_{ii} + O(\epsilon^2)$ no sum

$\sqrt{C_{ii}} = \sqrt{1 + 2\underbrace{\varepsilon_{ii}}_{x}} \approx 1 + \varepsilon_{ii} + O(\epsilon^2)$

Taylor series: $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$

$\Rightarrow \quad \varepsilon_{ii} \approx \sqrt{C_{ii}} - 1 = \lambda(\underline{e}_i) - 1$ ✓ *

$\lambda(\underline{dx}) = \dfrac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|}$ $\qquad \lambda(\underline{dx}) - 1 = \dfrac{|\underline{y} - \underline{x}| - |\underline{Y} - \underline{X}|}{|\underline{Y} - \underline{X}|}$

$dx = \underline{y} - \underline{x}$ $\qquad$ relative change in length

Off diagonal components

$$\sin\gamma(\underline{e}_i,\underline{e}_j) = \frac{C_{ij}}{\sqrt{C_{ii}}\,\sqrt{C_{jj}}} \qquad \text{(no sum)}$$

$$C_{ij} = 2\,\underline{\varepsilon}_{ij} + O(\epsilon^2) \qquad i \neq j$$

$$C_{ii} = 1 + 2\,\varepsilon_{ij} + O(\epsilon^2)$$

$$= 1 + O(\epsilon)$$

Marc does not understand

$$\sqrt{C_{ii}}\,\sqrt{C_{jj}} = (1+O(\epsilon))(1+O(\epsilon)) \simeq 1 + O(\epsilon^2)$$

ε_{ij}

$\epsilon \ll 1$

but if substitute

$$\sin\gamma(\underline{e}_i,\underline{e}_j) \approx \frac{2\,\varepsilon_{ij}}{1+} = 2\,\varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\sin\gamma(\underline{e}_i,\underline{e}_j) \qquad \text{if } \gamma \ll 1$$

$$\boxed{\varepsilon_{ij} \approx \frac{1}{2}\gamma(\underline{e}_i,\underline{e}_j)}$$

Linearization of kinematic quantities

$$\underline{\underline{H}} = \nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}}$$

Linearization of $\underline{\underline{U}}, \underline{\underline{V}}, \underline{\underline{R}}, \underline{\underline{C}}, \underline{\underline{E}}$ in limit

$\underline{\underline{H}} \ll 1$

Norm : $|\underline{\underline{H}}| = \sqrt{\underline{\underline{H}} : \underline{\underline{H}}} = (H_{11}^2 + H_{12}^2 + \dots + H_{33})^{1/2} = \epsilon \ll 1$

Taylor series: for any symmetric $\underline{\underline{A}}$ and $m \in \mathbb{R}$

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\,\underline{\underline{A}} + O(|A|^2) \dots \text{ as } |A| \to 0$$

Using this we can show

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + O(\epsilon^2)$$

$$\underline{\underline{V}}^2 = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{I}} + \underbrace{\underline{\underline{H}} + \underline{\underline{H}}^T}_{A} + O(\epsilon^2)$$

$$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + O(\epsilon^2)$$

$$\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + O(\epsilon^2)$$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) + O(\epsilon^2)$$

Identify two tensors:

$$\boxed{\begin{aligned} \underline{\underline{\varepsilon}} &= \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) = \mathrm{sym}(\nabla \underline{u}) \\ \underline{\underline{\omega}} &= \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) = \mathrm{skew}(\nabla \underline{u}) \end{aligned}}$$

infinit. stretch ten.

infinit. rotation tens

Example: $\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$

$$= (\underline{\underline{H}} + \underline{\underline{I}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}})^{-1}$$

$$= (\underline{\underline{H}} + \underline{\underline{I}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}} + O(\epsilon^2))$$

$$= \underline{\underline{H}} - \underbrace{\cancel{\underline{\underline{H}}\,\underline{\underline{\varepsilon}}}}_{\epsilon^2}{}^{\to 0} + \underline{\underline{I}} - \varepsilon + \cancel{O(\epsilon^2)}{}^{\to 0}$$

drop ϵ^2

$$\approx \underline{\underline{I}} + \underline{\underline{H}} - \tfrac{1}{2}(H + H^T)$$

$$\underline{\underline{R}} = \underline{\underline{I}} + \tfrac{1}{2}(H - H^{-T})$$

Infinitesimal strech & rotation decomp.:

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{H}} + \underline{\underline{I}} = \underline{\underline{I}} + \mathrm{sym}(\underline{\underline{H}}) + \mathrm{skew}(\underline{\underline{H}})$$

$$\boxed{\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}} \qquad |H| = \epsilon \ll 1$$

Large deformation: multiplicative decomposition

Small deformation: additive decomposition

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = (\underline{\underline{I}} + \underline{\underline{\omega}} + O(\epsilon^2))(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + O(\epsilon^2))$$

$$= I + \underline{\underline{\omega}} + \underline{\underline{\varepsilon}} + \underbrace{\underline{\underline{\omega}}\,\underline{\underline{\varepsilon}}}_{O(\epsilon^2)} + O(\epsilon^2)$$