Lecture 15: Motion & Material Time Derivative

Logistics: - PS 6 due today

Last time: - Infinitesimal strain tensor

$$\underline{u} = \varphi(\underline{X}) - \underline{X} \qquad \nabla\underline{u} = \underline{\underline{H}} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \text{sym}(\nabla\underline{u}) = \tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

$$\boxed{\underline{\underline{C}} = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \underbrace{\nabla\underline{u}^T\nabla\underline{u}}_{\text{non-linear}}}$$

$$\lim_{|\nabla\underline{u}|\to 0} \underline{\underline{E}} = \tfrac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \underline{\underline{\varepsilon}}$$

- $\varepsilon_{ii} \approx \underbrace{\lambda(\underline{e}_i) - 1}_{\text{rel. change in length}} \qquad \varepsilon_{ij} \approx \tfrac{1}{2}\underbrace{\sin\gamma(\underline{e}_i, \underline{e}_j)}_{\text{shear}} \approx \tfrac{1}{2}\gamma(\underline{e}_i, \underline{e}_j)$

- Rotation - Strech Decomposition

finite deformation: $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

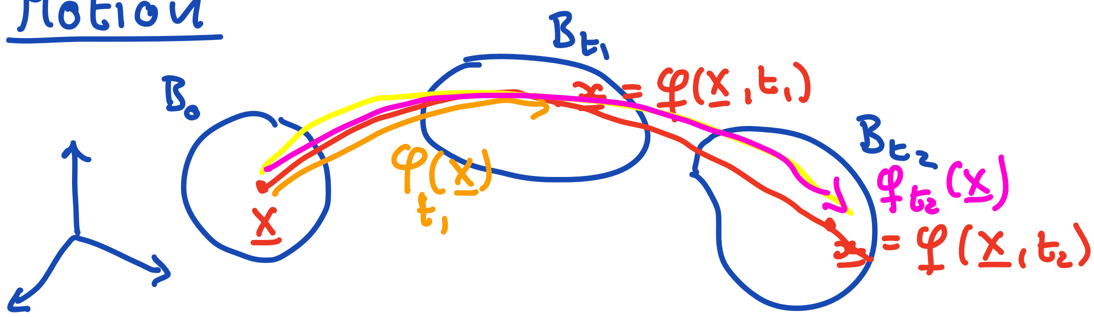
infinitesimal defor.: $\underline{\underline{F}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$

$\underline{\underline{\omega}} = \text{skew}(\nabla\underline{u})$

Today: Static $\rightarrow$ Dynamic deformation

$\Rightarrow$ Time derivatives

Motion



Motion is a <u>continuous</u> deformation of a body.

If motion is admissible $\Rightarrow$ inverse map

$$\boxed{\underline{X} = \psi(\underline{x}, t) = \varphi^{-1}(\underline{x}, t)}$$

<u>Material</u> field: naturally expressed in terms of $\underline{X}$

$$\Omega = \Omega(\underline{X}, t)$$

<u>Spatial</u> fields: naturally expressed in terms of $\underline{x}$

$$\Gamma = \Gamma(\underline{x}, t)$$

To any material field $\Omega(\underline{X}, t)$ we associate

a spatial field $\quad \Omega_s(\underline{x}, t) = \Omega(\psi(\underline{x}, t), t)$

we call Ω_s the <u>spatial description</u> of Ω

To any spatial field $\Gamma(\underline{x},t)$ we associate a material description : $\Gamma_m(\underline{X},t) = \Gamma(\varphi(\underline{X},t),t)$

Note: $\nabla_X \rightarrow$ derivative with respect to material coordinates

$\nabla_x \rightarrow$ derivative with respect to spatial coordinates

Velocity and acceleration fields

Naturally associated with particles $\Rightarrow$ material fields

Particle initially at $\underline{X}$ moving with motion

$\underline{x} = \varphi(\underline{X},t)$

$$\boxed{\begin{aligned} \text{velocity: } \quad & \underline{V}(\underline{X},t) = \frac{\partial}{\partial t}\varphi(\underline{X},t) = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{X}} \\ \text{acceleration: } \quad & \underline{A}(\underline{X},t) = \frac{\partial^2}{\partial t^2}\varphi(\underline{X},t) = \left.\frac{\partial^2 \underline{x}}{\partial t^2}\right|_{\underline{X}} \end{aligned}}$$

spatial descriptions are

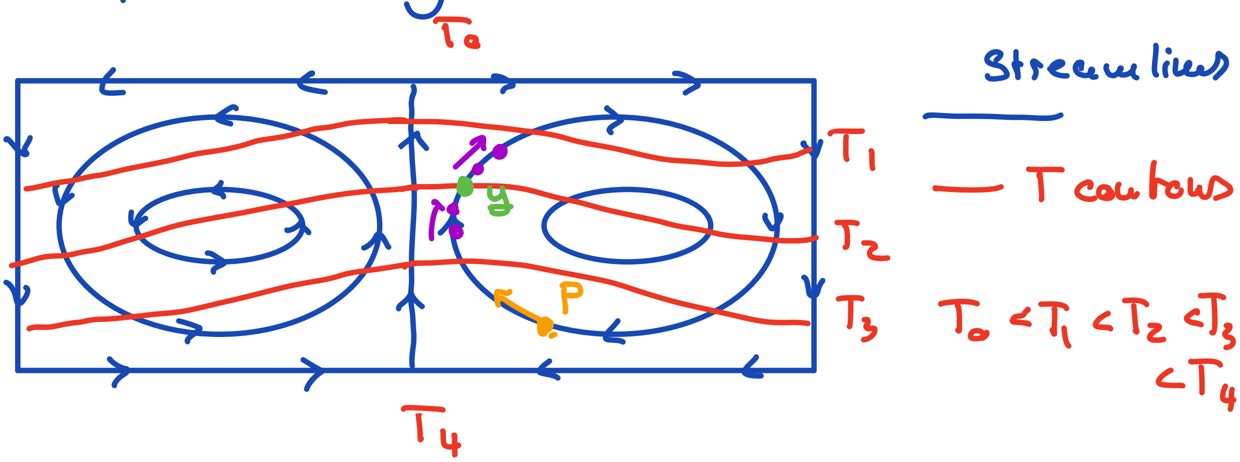
$$\underline{v}(\underline{x},t) = \underline{V}_s(\underline{x},t) = \frac{\partial}{\partial t}\underline{\varphi}(\underline{\Psi}(\underline{x},t),t)$$

$$\underline{a}(\underline{x},t) = \underline{A}_s(\underline{x},t) = \frac{\partial^2}{\partial t^2}\underline{\varphi}(\underline{\psi}(\underline{x},t),t)$$

These spatial fields correspond to the velocity and acceleration of the particle whose current coordinates are $\underline{x}$ and t.

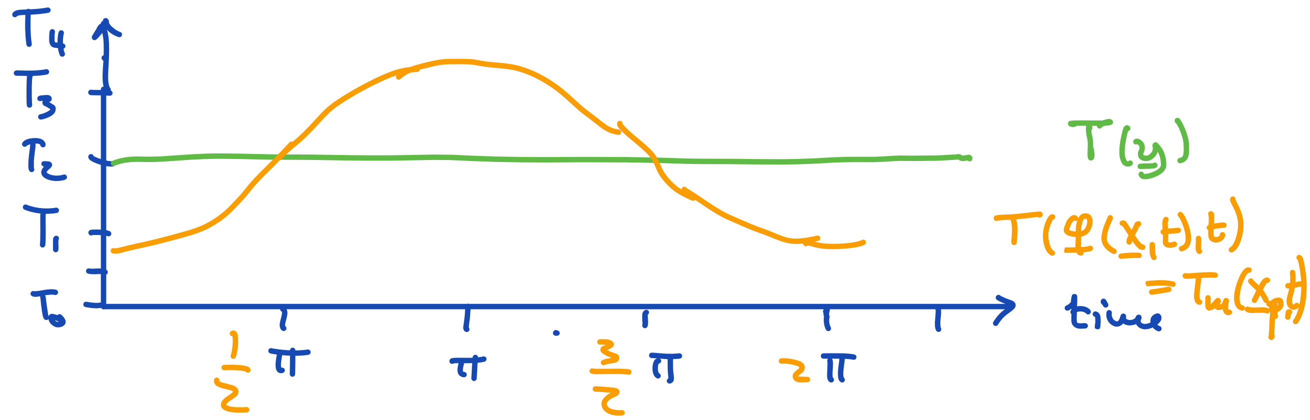
Note: Below we show that $\underline{a} \neq \frac{\partial \underline{v}}{\partial t}$

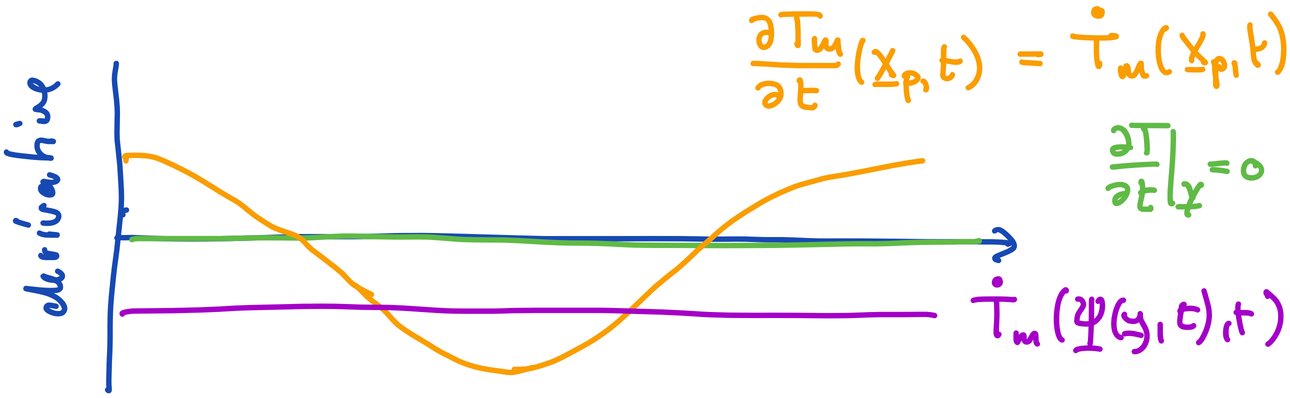
Example: Steady Convection



At steady state: $T(\underline{x},t) = T(\underline{x})$ spatial field

Particle P. with initial location $\underline{X}_p$





$\Rightarrow$ three types of time derivatives

Different time derivatives

I) Material time derivative of a material field Ω

$$\boxed{\dot{\Omega}(\underline{x},t) = \frac{D\Omega}{Dt}(\underline{x},t) = \left.\frac{\partial \Omega}{\partial t}\right|_{\underline{x}}}$$ (orange)

total, substantial, convective derivative

$\dot{\Omega}$ represents the rate of change of Ω as seen by an observer following the path line of a particle.

II) Spatial time derivative of spatial field Γ

Derivative with respect to t holding $\underline{x}$ fixed

$$\boxed{\frac{\partial \Gamma}{\partial t}(\underline{x}, t) = \frac{\partial \Gamma}{\partial t}\Big|_{\underline{x}}} \quad \text{(green)}$$

local time derivative

$\frac{\partial \Gamma}{\partial t}$ represents the rate of change of Γ as seen by an observer at $\underline{x}$

III) Material time derivative of a spatial field

Derivative of spatial field Γ with respect to time, hold $\underline{X}$ fixed. $\quad \underline{x} = \underline{\varphi}(\underline{X}, t)$

$$\boxed{\dot{\Gamma}(\underline{x}, t) = \frac{D\Gamma}{Dt}(\underline{x}, t) = \frac{\partial}{\partial t} \Gamma(\underbrace{\varphi(\underline{X}, t)}_{\underline{x}}, t)\Big|_{\underline{X} = \varphi(\underline{x}, t)}}$$

$$\underbrace{\Gamma(\varphi(\underline{X}, t), t)}_{\Gamma_m(t)}$$

$\Rightarrow$ two time dependencies

$$\frac{\partial}{\partial t}\Gamma_m(\underline{X},t) = \frac{\partial}{\partial t}\Gamma(\overbrace{\underline{\varphi}(\underline{X},t)}^{=},t) \qquad \text{chain rule}$$

$$= \frac{\partial \Gamma}{\partial t}(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)} + \frac{\partial \Gamma}{\partial x_i}(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)} \frac{\partial \varphi_i}{\partial t}(\underline{X},t)$$

recognize spatial velocity field: $v_i(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)} = \frac{\partial \varphi_i}{\partial t}(\underline{X},t)$

substitute

$$\frac{\partial \Gamma}{\partial t}(\underline{\varphi}(\underset{\uparrow}{\underline{X}},t),t) = \left[\frac{\partial \Gamma}{\partial t}(\underline{x},t) + \underbrace{\frac{\partial \Gamma}{\partial x_i}(\underline{x},t)\, v_i(\underline{x},t)}_{\nabla\Gamma \cdot \underline{v}}\right]\Bigg|_{\underline{x}=\varphi(\underline{X},t)}$$

material coordinates

Expressing result in spatial coordinates

$$\boxed{\dot{\Gamma}(\underline{x},t) = \frac{\partial \Gamma}{\partial t} + \nabla_x \Gamma \cdot \underline{v}}$$

This is important because it allows the computation of material derivative without knowledge of the motion $\underline{\varphi}$, if $\underline{v}$ is known

For a vector $\underline{\omega}$:

$$\boxed{\dot{\underline{\omega}} = \frac{\partial \underline{\omega}}{\partial t} + (\nabla_x \underline{\omega})\,\underline{v}}$$

In many books: $(\nabla_x \underline{\omega})\underline{v} = (\underline{v}\cdot\nabla)\underline{\omega}$ (HW5)

$$\boxed{\begin{aligned} \dot{T} &= \frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T \\ \dot{\underline{v}} &= \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} \end{aligned}}$$

scalar

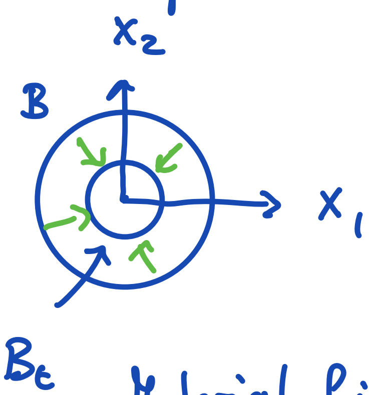
vector

ambiguous

The spatial acceleration field is

$$\boxed{\underline{a} = \dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v}}$$

Example: Exponential expansion/collapse



$$\underline{x} = \varphi(\underline{X},t) = e^{-\lambda t}\,\underline{X} \qquad \lambda \geq 0$$

$$\underline{X} = \Phi(\underline{x},t) = e^{\lambda t}\,\underline{x}$$

Material fields: $\underline{V}(\underline{X},t) = \frac{\partial}{\partial t}\varphi(\underline{X},t) = -\lambda\,\underline{X}\,e^{-\lambda t}$

$$\underline{A}(\underline{X},t) = \frac{\partial}{\partial t}\underline{V}(\underline{X},t) = \lambda^2\,\underline{X}\,e^{-\lambda t}$$

Spatial fields:

$$\underline{v}(\underline{x},t) = V_s(\underline{X},t) = \underline{V}(\Phi(\underline{x},t),t) = -\lambda e^{\lambda t}\,\underline{x}\,e^{-\lambda t}$$

$$= -\lambda\,\underline{x}$$

$$a(\underline{x},t) = \underline{A}(\Phi(\underline{x},t),t) = \lambda^2\,\underline{x}$$

Temperature field: $T_m(\underline{X},t) = \alpha\, t\, \|\underline{X}\|$

Material time derivative:

$$\dot{T}_m = \frac{\partial}{\partial t} T_m = \alpha \|\underline{X}\|$$

calculated directly from material description

Spatial T: $T(\underline{x},t) = T_m(\phi(\underline{x},t),t) = \alpha\, t\, \|\underline{x}\|\, e^{\lambda t}$

Suppose we only have spatial field

$T(\underline{x},t)$ and $\underline{v}(\underline{x},t)$

$$\dot{T}(\underline{x},t) = \frac{\partial T}{\partial t} + \nabla_x T \cdot \underline{v}$$

$$\frac{\partial T}{\partial t} = \alpha \|\underline{x}\| e^{\lambda t} + \alpha \lambda t \|\underline{x}\| e^{\lambda t}$$

$$\nabla_x T = \alpha\, t\, e^{\lambda t}\, \nabla_x (\underline{x}\cdot\underline{x})^{1/2} \qquad \nabla_x(\underline{x}\cdot\underline{x}) = 2\underline{x}$$

$$= \alpha\, t\, e^{\lambda t} \frac{\underline{x}}{\|\underline{x}\|}$$

Put it all together

$$\dot{T} = \underbrace{\alpha \|\underline{x}\| e^{\lambda t} + \alpha\lambda t \|\underline{x}\| e^{\lambda t}}_{\frac{\partial T}{\partial t}} + \underbrace{\alpha\, t\, e^{\lambda t} \frac{\underline{x}}{\|\underline{x}\|}}_{\nabla_x T} \cdot (-\lambda \underline{x})$$

$$= \alpha \|\underline{x}\| e^{\lambda t} + \cancel{\alpha \lambda t \|\underline{x}\| e^{\lambda t}} - \cancel{\alpha \lambda t e^{\lambda t}} \underbrace{\frac{\underline{x}\cdot\underline{x}}{\|\underline{x}\|}}_{\|\underline{x}\|}$$

$$\dot{T}(\underline{x},t) = \underline{\alpha \|\underline{x}\| e^{\lambda t}}$$

from above $\dot{T}_m(\underline{X}, t) = \alpha \|\underline{X}\|$

$$\dot{T} = \dot{T}_m(\psi(\underline{x}, t)) = \alpha \| e^{\lambda t} \underline{x} \| = \underline{\alpha \|x\| e^{\lambda t}}$$

$e^{\lambda t} \underline{x}$