Lecture 16: Rates

Logistics: - No HW this week (my apologies)

Last time: - Motions $x = \varphi(\underline{X}, t)$

- Material vs Spatial descripions

- 3 time derivatives:

1) Material time deriv. of material field:

$$\frac{\partial}{\partial t}\Omega(\underline{X}, t) = \frac{\partial \Omega}{\partial t}\bigg|_{\underline{X}} = \dot{\Omega}(\underline{X}, t)$$

2) Spatial time deriv. of spatial field

$$\frac{\partial}{\partial t}\Gamma(\underline{x}, t) = \frac{\partial \Gamma}{\partial t}\bigg|_{\underline{x}}$$

3) Material deriv. of spatial field

$$\dot{\Gamma}(\underline{x}, t) = \frac{\partial}{\partial t}\Gamma(\varphi(\underline{X}, t), t) = \boxed{\frac{\partial \Gamma}{\partial t} + \underline{v} \cdot \nabla \Gamma}$$

Today: - Rate of Deformation Tensors

- Reynolds transport theorem

- Derivatives of tensor function

$J = \det(\underline{\underline{F}})$

Rate of deformation tensor

→ similar role to deformation gradient but in rates

Velocity gradients

Spatial velocity grad:

$$\boxed{\underline{\underline{l}} = \nabla_x \underline{v}} \qquad l_{ij} = v_{i,j}$$

Material velocity gradient:

$$\underline{\underline{F}} = \nabla_X \underline{\varphi} \qquad F_{iJ} = \varphi_{i,J}$$

$$\underline{V} = \dot{\underline{\varphi}} \qquad V_i = \varphi_{i,t}$$

$$\Rightarrow \quad \dot{\underline{\underline{F}}} = \frac{\partial}{\partial t}(\nabla_X \underline{\varphi}) = \nabla_X \Big(\underbrace{\frac{\partial \underline{\varphi}}{\partial t}}_{\dot{\underline{\varphi}}}\Big) = \nabla_X \underline{V}$$

$$\boxed{\dot{\underline{\underline{F}}} = \nabla_X \underline{V}}$$

Note analogy:

$$\underline{\varphi}(\underline{X} + \underline{\Delta X}, t) \approx \underline{\varphi}(\underline{X}, t) + \underbrace{\underline{\underline{F}}}_{\nabla \varphi}(\underline{X}, t)\, \Delta X \qquad \text{T.S.}$$

take material derivative

$$\dot{\underline{\varphi}} = V(\underline{X} + \Delta X, t) \approx \underbrace{\dot{\underline{\varphi}}(X, t)}_{V(X,t)} + \underbrace{\dot{\underline{\underline{F}}}(\underline{X}, t)}_{\nabla_X V}\, \underline{\Delta X}$$

Def: $\boxed{\begin{aligned}&\varphi(\underline{X}+\Delta\underline{X},t) \approx \varphi(\underline{X},t) + \nabla_X\varphi\,\Delta X\\ \text{Vel:}\quad &\underline{V}(\underline{X}+\Delta\underline{X},t) \approx \underline{V}(\underline{X},t) + \nabla_X V\,\Delta X\end{aligned}}$

Note: $\underline{V}(\underline{X},t) = v(\varphi(\underline{X},t),t)$

$$\nabla_X V \neq \nabla_x \underline{v}\,\big|_{\underline{x}=\varphi(\underline{X},t)}$$

derivatives are in different directions

To relate $\nabla_x \underline{v}$ and $\nabla_X \underline{V}$ use

$$\underline{V}(\underline{X},t) = \underline{v}(\varphi(\underline{X},t),t)$$

$$\dot{F}_{iJ} = \frac{\partial}{\partial X_J} V_i = \underbrace{\frac{\partial}{\partial X_J}}\, v_i(\varphi(\underline{X},t),t)$$

$$\frac{\partial}{\partial X_J} = \frac{\partial}{\partial x_k}\underbrace{\frac{\partial x_k}{\partial X_J}}_{\varphi_{kJ} = F_{kJ}} = \frac{\partial}{\partial x_k} F_{kJ}$$

substitute

$$\dot{F}_{iJ} = \frac{\partial}{\partial X_J} v_i(\varphi(\underline{X},t),t) = \frac{\partial}{\partial x_k} v_i\, F_{kJ}$$

$\underline{F}^{-1}$

$$\dot{F}_{iJ} = v_{i,k} F_{kJ} \quad \Rightarrow \quad \dot{\underline{\underline{F}}} = \nabla_x \underline{v}\ \underline{\underline{F}} = \underline{\underline{\ell}}\ \underline{\underline{F}}$$

$$\Rightarrow \quad \boxed{\nabla_X \underline{V} = \nabla_x \underline{v}\ \underline{\underline{F}}}$$

$$\boxed{\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\ \underline{\underline{F}}^{-1}}$$

To understand $\underline{\underline{\ell}} = \nabla_x \underline{v}$ need to decompose

similar to $\underline{E} = \nabla \varphi$ and $\underline{\underline{H}} = \nabla \underline{u}$

finite strain: $\underline{\underline{F}} = \underline{\underline{R}}\ \underline{\underline{U}}$

infinitesimal strain: $\underline{\underline{H}} = \mathrm{sym}(\underline{\underline{H}}) + \mathrm{skew}(\underline{\underline{H}})$

Decomposition of $\underline{\underline{\ell}}$

$$\boxed{\begin{aligned} \underline{\underline{\ell}} &= \underline{\underline{d}} + \underline{\underline{w}} \\ \underline{\underline{d}} &= \mathrm{sym}(\underline{\underline{\ell}}) = \tfrac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right) \\ \underline{\underline{w}} &= \mathrm{skew}(\underline{\underline{\ell}}) = \tfrac{1}{2}\left(\nabla_x \underline{v} - \nabla_x \underline{v}^T\right) \end{aligned}}$$

rate of strain tensor

spin tensor

Interpretation

$$\underline{v}(\underline{x}+\Delta\underline{x},t) \approx \underline{v}(\underline{x},t) + \underbrace{\nabla_x \underline{v}}_{\underline{\underline{l}}=\underline{\underline{d}}+\underline{\underline{w}}} \Delta\underline{x}$$

$$\approx \underline{v}(\underline{x},t) + \underline{\underline{d}}\,\Delta\underline{x} + \underline{\underline{w}}\,\Delta\underline{x}$$

because $\underline{\underline{w}}$ is skew $\Rightarrow$ axial vector $\underline{\omega} = \text{vec}(\underline{\underline{w}})$

so that $\underline{\underline{w}}\,\Delta x = \underline{\omega} \times \Delta\underline{x}$

$$\underline{v}(\underline{x}+\Delta\underline{x},t) = \underline{v}(\underline{x},t) + \underline{\underline{d}}\,\Delta\underline{x} + \underline{\omega} \times \Delta\underline{x}$$

$\Rightarrow$ $\underline{\underline{d}}$ is rate of change of shape (stretch rate)

$\underline{\underline{w}}$ is rate of change in orientation (spin)

where $|\underline{\omega}|$ is the angular velocity

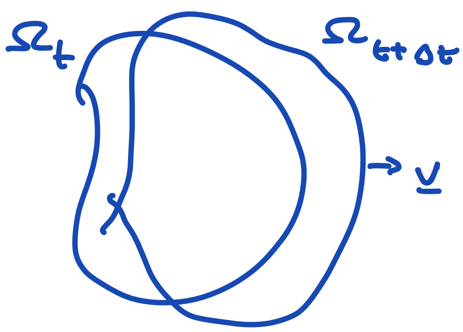
$\Rightarrow$ vorticity: $\nabla_x \times \underline{v} = 2\underline{\omega}$

By analogy to infinitesimal deformation the diagonal components of $\underline{\underline{d}} = \text{sym}(\nabla_x \underline{v})$ quantify instantaneous rate of stretching in $\underline{e}_i$ directions. off diagonal components of $\underline{\underline{\varepsilon}}$ quantify inst. rate of shear between coord. dir.

Reynolds Transport Theorem

motion $\underline{\varphi}(\underline{X}, t)$ with spatial velocity field $\underline{v}(\underline{x}, t)$

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t} dV_x + \oint_{\partial\Omega_t} \phi \, \underline{v}\cdot\underline{n} \, dA_x}$$



Multi D analog to Leibitz rule for integrals with boundaries that change

Difficulty is that Ω_t changes with time.

To prove RTT move to reference config. Ω_0

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t) \, \underset{\underset{J\,dV_X}{\uparrow}}{dV_x}} = \frac{d}{dt}\int_{\Omega_0} \underbrace{\phi(\underline{\varphi}(\underline{X},t),t)}_{\phi_m(\underline{X},t)} J(\underline{X},t) \, dV_X$$

$$= \frac{d}{dt} \int_{\Omega_0} \phi_m(\underline{X}, t)\, J(\underline{X}, t)\, dV_X$$

Because Ω_0 is fixed $\Rightarrow$ exchange deriv. & integr.

$$= \int_{\Omega_0} \frac{d}{dt} \phi_m(\underline{X}, t)\, J(\underline{X}, t)\, dV_X$$

$$= \int_{\Omega_0} (\dot{\phi}_m J + \phi_m \dot{J})\, dV_X$$

show later: $\boxed{\dot{J} = J\,(\nabla_x \cdot \underline{v})_m}$

$$= \int_{\Omega_0} \dot{\phi}_m J + \phi_m J\,(\nabla_x \cdot \underline{v})_m\, dV_X$$

$$= \int_{\Omega_0} (\dot{\phi}_m + \phi_m (\nabla_x \cdot \underline{v})_m)\, \underbrace{J\, dV_X}_{dV_x}$$

$$= \int_{\Omega_t} \dot{\phi} + \phi\, \nabla_x \cdot \underline{v}\;\; dV_x$$

where: $\dot{\phi} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla_x \phi$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \underbrace{\underline{v} \cdot \nabla_x \phi + \phi\, \nabla_x \cdot \underline{v}}_{\nabla \cdot (\phi \underline{v})}\; dV_x$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v}) \, dV_x$$

use div. thm $\int_{\Omega_t} \nabla \cdot (\phi \underline{v}) \, dV_x = \oint_{\partial \Omega_t} \phi \underline{v} \cdot \underline{n} \, dA_x$

$$\frac{d}{dt} \int_{\Omega_t} \phi(\underline{x}, t) \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t} \, dV_x + \oint_{\partial \Omega_t} \phi \underline{v} \cdot \underline{n} \, dA_x \quad \checkmark$$

Derivative of a tensor function

So far we have considered fields:

$$\phi(\underline{x}), \quad \underline{v}(\underline{x}), \quad \underline{\underline{S}}(\underline{x})$$

Now we are interested in tensor functions

- scalar valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor-valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar valued tensor functions

Typical examples: $\det(\underline{\underline{A}})$ or $tr(\underline{\underline{A}})$

Def: $\psi(\underline{\underline{S}})$ is differentiable at $\underline{\underline{A}}$ if

there exists a tensor $D\psi(\underline{\underline{A}})$ s.t.

$$\psi(\underline{\underline{A}}+\underline{\underline{H}}) = \psi(\underline{\underline{A}}) + D\psi(\underline{\underline{A}}) : \underline{\underline{H}} + h.o.t.$$

or equivalently $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$ $\quad |\underline{\underline{U}}| = \sqrt{\underline{\underline{U}}:\underline{\underline{U}}} = 1$

$$\boxed{D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon=0}}$$

$D\psi(\underline{\underline{A}})$ is called the derivative of ψ at $\underline{\underline{A}}$

In frame $\{\underline{e}_i\}$ we have

$$\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j}$$

To see this $\psi(A_{11}, A_{12}, \dots, A_{33})$

and $\underline{\underline{U}} = U_{kl}\, \underline{e}_k \otimes \underline{e}_l$

$$\psi(\underbrace{\underline{\underline{\bar{A}}} + \epsilon \underline{\underline{U}}}_{\underline{\underline{A}}}) = \psi(\bar{A}_{11} + \epsilon U_{11}, \bar{A}_{12} + \epsilon U_{12}, \dots \bar{A}_{33} + \epsilon U_{33})$$

$$D\psi(\underline{\underline{A}}) : \underline{\underline{u}} = \frac{d}{d\epsilon}\psi(\underbrace{\bar{A}_{11}+\epsilon u_{11}}_{A_{11}}, \bar{A}_{12}+\epsilon u_{12} \dots)\Big|_{\epsilon=0}$$

$$= \frac{\partial\psi}{\partial A_{11}} u_{11} + \frac{\partial\psi}{\partial A_{12}} u_{12} + \dots + \frac{\partial\psi}{\partial A_{33}} u_{33}$$

$$=$$

$$= \frac{\partial\psi}{\partial A_{ij}} u_{ij} = \left(\frac{\partial\psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j\right) : \left(u_{kl}\, \underline{e}_k \otimes \underline{e}_l\right)$$

by arbitrary ness of $\underline{\underline{u}}$

$$\Rightarrow \quad \underline{\underline{D}}\psi(\underline{\underline{A}}) = \frac{\partial\psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j$$

Derivative of trace

$$\psi(\underline{\underline{A}}) = tr(\underline{\underline{A}}) = A_{ii}$$

$$D\,tr(\underline{\underline{A}}) = \frac{\partial tr(\underline{\underline{A}})}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \underbrace{\frac{\partial A_{ii}}{\partial A_{kl}}}_{\delta_{ik}\delta_{il}} \underline{e}_k \otimes \underline{e}_l$$

$$= \delta_{ik}\delta_{il}\, \underline{e}_k \otimes \underline{e}_l$$

$$= \underline{e}_i \otimes \underline{e}_i = \sum_{i=1}^{3} \underline{e}_i \otimes \underline{e}_i = \underline{\underline{I}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad D\,\mathrm{tr}(\underline{\underline{A}}) = \underline{\underline{I}}$$

Derivative of the determinant

$\varphi(\underline{\underline{A}}) = \det(\underline{\underline{A}})$ if $\underline{\underline{A}}$ is invertible

$$\boxed{D\det(\underline{\underline{A}}) = \det(\underline{\underline{A}})\,\underline{\underline{A}}^{-T}}$$

from the def. of dir. deriv.

$$D\det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) = \frac{d}{d\epsilon} \det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon=0}$$

Simplify expansion

$$\det(\epsilon \underline{\underline{U}} + \underline{\underline{A}}) = \det\left(\epsilon \underline{\underline{A}}\left(\underline{\underline{A}}^{-1}\underline{\underline{U}} + \frac{1}{\epsilon}\underline{\underline{I}}\right)\right) \qquad \frac{1}{\epsilon} = -\lambda$$

$$= \det(\epsilon \underline{\underline{A}}) \det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda \underline{\underline{I}})$$

$$= \epsilon^3 \det(\underline{\underline{A}}) \det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda \underline{\underline{I}})$$

from. def. of principal invariants

$$\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda \underline{\underline{I}}) = -\lambda^3 + \lambda^2 I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) - \lambda I_2(\underline{\underline{A}}^{-1}\underline{\underline{U}}) + I_3(\underline{\underline{A}}^{-1}\underline{\underline{U}})$$

$$= -\left(-\frac{1}{\varepsilon}\right)^3 + \left(-\frac{1}{\epsilon}\right)^2 I_1 + \frac{1}{\varepsilon} I_2 + I_3$$

$= \frac{1}{[illegible]} + \frac{1}{[illegible]} I_{[illegible]} + \frac{1}{[illegible]} I_{[illegible]} + I_{[illegible]}$

$\underbrace{\epsilon^3 \qquad \epsilon^2 \; {}^{-1} \qquad \epsilon \; {}^{-2} \qquad {}^{-3}}$

substitute expansion above

$$\det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) = \epsilon^3 \det(\underline{\underline{A}}) \left(\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2} I_1 + \frac{1}{\epsilon} I_2 + I_3 \right)$$
$$= \det(\underline{\underline{A}}) \left(1 + \epsilon I_1 + \epsilon^2 I_2 + \epsilon^3 I_3 \right)$$

$$D \det(A) : \underline{\underline{U}} = \frac{d}{d\epsilon} \det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon=0}$$
$$= \det(\underline{\underline{A}}) \frac{d}{d\epsilon} \left(1 + \epsilon I_1 + \epsilon^2 I_2 + \epsilon^3 I_3 \right) \Big|_{\epsilon=0}$$

$$= \det(\underline{\underline{A}}) \left(I_1 + 2\epsilon I_2 + 3\epsilon^2 I_3 \right) \Big|_{\epsilon=0}$$
$$= \det(\underline{\underline{A}}) I_1 (\underline{\underline{A}}^{-1} \underline{\underline{U}})$$

$$I_1 (\underline{\underline{A}}^{-1} \underline{\underline{U}}) = \mathrm{tr} (\underline{\underline{A}}^{-1} \underline{\underline{U}}) = \mathrm{tr} (A^{-1}_{ij} U_{jk} \, \underline{e}_i \otimes \underline{e}_k)$$
$$= A^{-1}_{ij} U_{ji} = A^{-T}_{ji} U_{ji} = \underline{\underline{A}}^{-T} : \underline{\underline{U}}$$

$$\underline{D \det(\underline{\underline{A}})} : \underline{\underline{U}} = \underline{\det(\underline{\underline{A}}) \, \underline{\underline{A}}^{-T}} : \underline{\underline{U}}$$

Time derivative

$$\underline{\underline{S}} = \underline{\underline{S}}(t)$$

$$\dot{\underline{\underline{S}}} = \frac{d\underline{\underline{S}}}{dt} = \frac{d\dot{S}_{ij}}{dt}\, \underline{e}_i \otimes e_j$$

$$\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$$

Jacobi's formula: $\boxed{\frac{d}{dt}\det(\underline{\underline{S}}(t)) = \det(\underline{\underline{S}})\, \underline{\underline{S}}^{-T} : \dot{\underline{\underline{S}}}}$

$\det(F)$

$$\dot{J} = J\, \underline{\underline{F}}^{-T} : \dot{F}$$