Lecture 17: Balance Laws in integral form

Logistics: - HW7 has been posted due next week

- next week office hours $\rightarrow$ project discussion

Last time: Rate of deformation tensor

$$\nabla_x \underline{V} = \dot{\underline{\underline{F}}} \qquad \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\, \underline{\underline{F}}^{-1}$$

$$\underline{\underline{d}} = \mathrm{sym}(\nabla_x \underline{v}) = \frac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right) \quad \text{rate of strain tensor}$$

$$\underline{\underline{\omega}} = \mathrm{skew}(\nabla_x \underline{v}) = \frac{1}{2}\left(\nabla_x \underline{v} - \nabla_x \underline{v}^T\right) \quad \text{spin tensor}$$

Reynolds transport theorem:

$$\frac{d}{dt}\int_{\Omega_t} \phi \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t}\, dV_x + \oint_{\partial \Omega_t} \phi\, \underline{v} \cdot \underline{n}\, dA_x$$

Derivatives of tensor functions

$$\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}}\, \underline{e}_i \otimes \underline{e}_j}$$

Time derivative: $\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$

Today: From particles $\rightarrow$ continuum

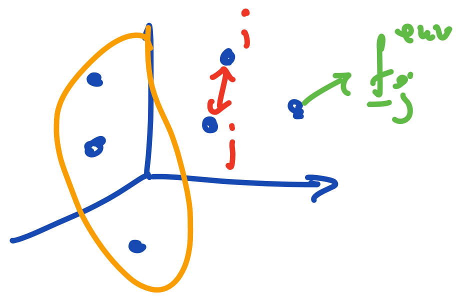
Energy, work, power

Laws of inertia

Continuum Thermodynamics

Balance laws

System of N particles



m_i masses

$\underline{x}_i$ locations

$U_{ij} = U_{ji}$ interaction energy

$\underline{f}_{ij}^{int} = -\nabla_x U_{ij}$ interaction force

$\underline{f}_i^{env}$ = environmental force

Mass is constant: $\dot{m}_i = 0$

Newtons 2^{nd} law: $m_i \ddot{\underline{x}}_i = \underline{f}_i^{env} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \underline{f}_{ij}^{int}$ $\Big\}$ $i = 1 \dots N$

For any subset Ω of particles $I = \{3, 7, 256..$

total mass: $M[\Omega] = \sum_{i\in I} m_i$

linear mom.: $\underline{L}[\Omega] = \sum_{i\in I} m_i \dot{\underline{x}}_i$

angular mom.: $\underline{j}[\Omega] = \sum_{i\in I} \underline{x}_i \times m_i \dot{\underline{x}}_i$

kinetic energy: $\overset{K}{\cancel{U}}[\Omega] = \sum_{i \in I} \frac{1}{2} m_i |\dot{\underline{x}}_i|^2$

internal energy: $U[\Omega] = \sum_{\substack{i,j \in I \\ i > j}} U_{ij}$

Balance laws:

Mass is conserved: $\frac{d}{dt} M[\Omega] = \dot{M}[\Omega] = 0$

$$\frac{d}{dt} \underline{L}[\Omega] = \sum_{i \in I} \left[\underline{f}_i^{env} + \sum_{j \neq i} \underline{f}_{ij}^{int} \right]$$

$$\frac{d}{dt} \underline{j}[\Omega] = \sum_{i \in I} \underline{x}_i \times \left[\underline{f}_i^{env} + \sum_{j \neq i} \underline{f}_{ij}^{int} \right]$$

Change in total (internal + kinetic) energy is due power of external forces

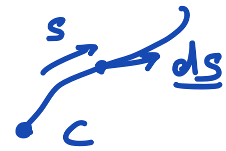
$$\frac{d}{dt} \left(U[\Omega] + K[\Omega] \right) = \sum_{i \in I} \underbrace{\dot{\underline{x}}_i}_{\underline{v}} \cdot \underbrace{\left[\underline{f}_i^{env} + \sum_{j \neq i} \underline{f}_{ij} \right]}_{f}$$

Reminder:

Work = energy transferred by application of force along a distance

$W = f\, s$ $\quad$ s = distance $\quad$ f = mag. force

$$W = \int_C \underline{f} \cdot d\underline{s} = \int_{t_1}^{t_2} \underline{f} \cdot \underbrace{\frac{d\underline{s}}{dt}}_{\underline{v}}\, dt$$



$$W = \int_{t_1}^{t_2} \left(\underline{f} \cdot \underline{v}\right) dt = \int \left(\frac{dW}{dt}\right) dt$$

Power is rate of work

$$\boxed{P = \frac{dW}{dt} = \underline{f} \cdot \underline{v}}$$

Generalize discrete $\rightarrow$ continuum : $\Sigma \rightarrow \int$

$\Rightarrow$ mass, lin. & ang. mom.

Continuum energy balance is more complicated because we loose inform. about velocity fluctuations

Continuum we only have mean velocity

$\Rightarrow$ Introduce new variables

Temperature: measure of magnitude of velocity fluctuations

Heat - measure of energy fluctuations

$\Rightarrow$ Continuum Thermodynamics

first lets deal with mass & mom.

Balance laws in integral form

$$\sum m_i \rightarrow \int \rho \, dV$$

$$M[\Omega] = \int_\Omega \rho(\underline{x},t) \, dV_x \qquad \text{mass}$$

$$\underline{L}[\Omega] = \int_\Omega \rho(\underline{x},t) \underline{v}(\underline{x},t) \, dV_x \qquad \text{lin. mom.}$$

$$\underline{j}[\Omega]_z = \int_\Omega (\underline{x}-\underline{z}) \times \rho(\underline{x},t) \underline{v}(\underline{x},t) \, dV_x \qquad \text{ang. mom.}$$

Conservation of mass

In absence of reactions, relativistic effects or radioactive decay the mass of a continuum body does not change:

$$\boxed{\frac{d}{dt} M[\Omega] = 0}$$

Laws of inertia

In fixed frame of reference, the rate of change of lin. & ang. momentum in Ω are equal to the resultant force & torque.

$$\boxed{\begin{aligned} \frac{d}{dt}\underline{L}[\Omega] &= \int_\Omega \rho \underline{b}\, dV_x + \int_{\partial\Omega} \underline{t}\, dA \\ \frac{d}{dt}\underline{J}[\Omega] &= \int_\Omega \underline{x} \times \rho \underline{b}\, dV_x + \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA \end{aligned}}$$

Continuum Thermodynamics

Temperature and heat:

assume existence of absolute Temperature field $\theta(\underline{x}, t) \geq 0$ that is measure of the velocity fluctuations of atoms in vicinity of $\underline{x}$.

Thermal energy or heat content is the energy associated with velocity fluctuations.

Bodies can exchange heat & mechanical work.

Heat can be gained/lost in 2 ways:

I) Rate of body heating: $Q_b[\Omega] = \int \rho \, r \, dV_x$

II) Rate of surface heating: $Q_s[\Omega] = -\int_{\partial\Omega} \underline{q} \cdot \underline{n} \, dA_x$

$r(\underline{x}, t)$ is heat prod./loss per unit mass

$\underline{q}(\underline{x}, t)$ is heat flux vector

Net rate of heating

$$\boxed{Q[\Omega] = Q_b[\Omega] + Q_s[\Omega] = \int_{\Omega} \rho \, r \, dV_x - \int_{\partial\Omega} \underline{q} \cdot \underline{n} \, dA_x}$$

Kinetic Energy of continuum body

$$K[\Omega] = \int_{\Omega} \frac{1}{2} \rho |\underline{v}|^2 dV_x$$

Power of external forces acting on Ω

$$P[\Omega] = \int_{\Omega} \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega} \underline{t} \cdot \underline{v} \, dA_x$$

Rate net working: $W[\Omega]$ of external forces is the mechanical power not converted into motion

$$\mathbb{W}[\Omega] = \mathbb{P}[\Omega] - \frac{d}{dt} K[\Omega]$$

$\mathbb{W}[\Omega] > 0$: mechanical energy

$\mathbb{W}[\Omega] < 0$: stored energy is released

Internal Energy and the First law

Energy not associated with kinetic energy is called internal energy.

We assume that internal energy consists only of thermal (heat) and elastic (mech.) energy.

$$U[\Omega] = \int_\Omega \rho \, u \, dV_x$$

$u =$ internal energy density per unit mass

First law of thermodynamics

$$\boxed{\frac{d}{dt} U[\Omega] = Q[\Omega] + \mathbb{W}[\Omega]}$$

$dU = dQ + dW$

or

$$\frac{d}{dt}\left(U[\Omega] + K[\Omega]\right) = Q[\Omega] + P[\Omega]$$

In some cases the power of an external force can be written: $P[\Omega] = -\frac{d}{dt} G[\Omega]$

where $G[\Omega]$ is called potential energy

$$\Rightarrow \quad \frac{d}{dt}\Big(\underset{\text{internal}}{U[\Omega]} + \underset{\text{kinetic}}{K[\Omega]} + \underset{\substack{\uparrow \\ \text{grav. pot.}}}{G[\Omega]}\Big) = Q[\Omega]$$

Entropy and Second Law

The 2nd law expresses the fact that a body has a limit on the rate of heat uptake, but has no limit on the rate of heat release.

The 2nd law postulates:

$$Q[\Omega] \leq \Xi\,[\Omega] \qquad \text{"capital xi"} \quad \xi \ \Xi$$

upper bound on rate of net heating

In the absence of work $W[\Omega]=0$

$$\frac{d}{dt}U[\Omega] = Q[\Omega] \leq \Xi[\Omega]$$

$\Rightarrow$ limits the rate of heat storage

The entropy of a body is defined (up to a const.)

$$\boxed{\frac{d}{dt}S[\Omega] = \frac{\Xi[\Omega]}{\Theta[\Omega]}}$$

irreversible

Entropy is the quantity whose rate of change is equal to the upper heating bound per unit temp.

Entropy in terms of net heating

$$\boxed{\frac{d}{dt}S[\Omega] \geq \frac{Q[\Omega]}{\Theta[\Omega]}}$$

Clausius-Planck relation

reversible

In thermo books: $dS = \frac{dQ_{rev}}{T}$ and $dS \geq \frac{dQ}{T}$

The irreversibility of natural processes is shown by $\frac{d}{dt}S[\Omega] \geq 0$ if $Q[\Omega]=0$

For non homogeneous bodies we introduce entropy density field, $s(x,t)$, so that

$$S[\Omega] = \int_{\Omega} \rho\, s \, dV_x$$

Generalization of 2^{nd} law to continous systems is given by Clausius-Duhem Equ:

$$\boxed{\frac{d}{dt}\int_{\Omega} \rho\, s \, dV_x \geq \int_{\Omega} \frac{\rho\, r}{\theta} \, dV_x - \int_{\partial\Omega} \frac{q \cdot n}{\theta} \, dA_x}$$

This relation places restrictions on constitutive relations and leads to statements about energy dissipation.