Lecture 18: Local Eulerian Balance laws

Logistics: HW7 is due next Th.

Graduate students please come to office hours next week

→ google sheet "sign up"

Last time: - Balance laws for discrete particles

= Integral balance laws $\Sigma \to \int$

loose info about velocity fluctuations

⇒ new variables: T, Q

- Continuum thermo:

  - Net rate of heating: $Q = Q_b + Q_s$
  - Net rate of working: $W = P - \frac{dK}{dt}$

- First law: $\frac{dU}{dt} = Q + W$

- Second law: $\frac{dS}{dt} \geq \frac{Q}{\Theta}$

Today: Local Eulerian Balance Laws spatial

- mass, linear & ang. momentum.

Local Eulerian Balance Laws

motion $\quad x = \varphi(X, t)$

some arbitrary subdomain Ω

$$\Omega_t = \varphi(\Omega, t)$$

I Conservation of mass

Integral form: $\quad \frac{d}{dt} M[\Omega_t] = 0 \quad \Rightarrow M[\Omega] = M[\Omega_t]$

$$M[\Omega_t] = \int_{\Omega_t} \rho(x,t)\, dV_x = \int_{\Omega} \rho_m(X,t)\, J(X,t)\, dV_X$$

where $\quad J(X,t) = \det(F(X,t)) \qquad J = \frac{dV_x}{dV_X}$

At time $t = 0 \qquad x = X \qquad \Omega_t = \Omega \qquad \det F = 1$

$$M[\Omega] = \int_{\Omega_0} \rho(x,0)\, dV_x = \int_{\Omega} \underbrace{\rho_m(X,0)}_{\rho_0(X)}\, dV_X = \int \rho_0(X)\, dV_X$$

Conservation of mass

$$M[\Omega_t] - M[\Omega_0] = \int_{\Omega} \underbrace{\left[\rho_m(X,t)\, J(X,t) - \rho_0(X)\right]}_{0} dV_X = 0$$

because Ω is arbitrary

$$\Rightarrow \quad \boxed{\rho_m(X,t)\, J(X,t) = \rho_0(X)}$$

Lagrangian statement of mass conservation

$$\rho_m(\underline{X},t)\, dV_x = \rho_0(\underline{X})\, dV_X$$

To convert this to Eulerian form we take $\frac{\partial}{\partial t}$

$$\underbrace{\frac{\partial}{\partial t}\rho_m(\underline{X},t)}_{\dot{\rho}(\underline{x},t)} \cancel{J(\underline{X},t)} + \rho_m(\underline{X},t) \underbrace{\dot{J}(\underline{X},t)}_{\cancel{J(X,t)}\,(\nabla_x \cdot \underline{v})_m} = 0$$

dividing by J and switching to spatial description

$$\dot{\rho}(\underline{x},t) + \rho(\underline{x},t)\nabla_x \cdot \underline{v} = 0$$

$$\boxed{\dot{\rho} + \rho \nabla_x \cdot \underline{v} = 0}$$ local Eulerian form mass balance

expanding ~~wed~~ $\dot{\rho} = \frac{\partial \rho}{\partial t} + \nabla_x \rho \cdot \underline{v}$

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla_x \rho \cdot \underline{v} + \rho\, \nabla_x \cdot \underline{v}}_{\nabla_x \cdot (\rho \underline{v})} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \underline{v}) = 0}$$ conservative local eulerian mass balance

Time derivative of integrals relative to mass

$$\boxed{\frac{d}{dt}\int_{\Omega_t}\phi(\underline{x},t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega}\dot{\phi}(\underline{x},t)\,\rho(\underline{x},t)\,dV_x}$$

where $\phi(\underline{x},t)$ is any scalar, vector or tensor field

$$\int_{\Omega_t}\phi(x,t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega}\phi_m(\underline{X},t)\,\underbrace{\rho_m(\underline{X},t)\,J(\underline{X},t)}_{\rho_0(\underline{X})}\,dV_X$$

$$= \int_{\Omega}\phi_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X$$

Take time derivative

$$\frac{d}{dt}\int_{\Omega_t}\phi(\underline{x},t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega}\dot{\phi}_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X$$

$$= \int_{\Omega}\dot{\phi}_m(\underline{X},t)\,\rho_m(\underline{X},t)\,\underbrace{J(\underline{X},t)\,dV_X}_{dV_x}$$

$$= \int_{\Omega_t}\dot{\phi}(\underline{x},t)\,\rho(\underline{x},t)\,dV_x$$

Balance of linear momentum

$$\frac{d\underline{L}}{dt} = \underline{F}_b + \underline{F}_s \qquad \underline{L} = \int \rho\underline{v}\,dV$$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{t}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

Cauchy stress: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

tensor divergence theorem

$$\int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\, dA_x = \int_{\Omega_t} \nabla\cdot\underline{\underline{\sigma}}\, dV_x$$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\Omega_t} \rho \dot{\underline{v}}\, dV_x$$

substitute

$$\int_{\Omega_t} [\rho\dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} - \rho\underline{b}]\, dV_x = \underline{0}$$

because Ω is arbitrary $\rightarrow$ integrand $= 0$

$$\boxed{\rho\dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} = \rho\underline{b}}$$ local Eulerian lin. mom. bal.

$\rho\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} - \nabla\cdot\underline{\underline{\sigma}} = \rho\underline{b}$ Cauchy's first equation of motion

To rewrite in conservative form

$$\rho\dot{\underline{v}} = \rho\left(\frac{\partial \underline{v}}{\partial t} + \underbrace{(\nabla_x \underline{v})\underline{v}}_{(\underline{v}\cdot\nabla)\underline{v}}\right) = \rho\frac{\partial \underline{v}}{\partial t} + \rho(\nabla_x \underline{v})\underline{v}$$

$\rho\frac{\partial \underline{v}}{\partial t}$

$$= \frac{\partial}{\partial t}(\rho \underline{v}) - \underline{v}\underbrace{\frac{\partial \rho}{\partial t}}_{-\nabla\cdot(\rho \underline{v}) \text{ from mass bal.}} + \rho(\nabla_x \underline{v})\,\underline{v}$$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \boxed{\nabla_x \cdot (\rho \underline{v})\,\underline{v} + (\nabla_x \underline{v})(\rho \underline{v})}$$

use $\nabla \cdot (\underline{a} \otimes \underline{b}) = (\nabla \underline{a})\,\underline{b} + \underline{a} \cdot \nabla \cdot \underline{b}$ (HW5 Q4)

$\underline{b} = \rho \underline{v} \qquad \underline{a} = \underline{v}$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v})$$

$\Rightarrow$ local conservative Eulerian lin. mom. bal.

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

body force gravity

conserved quantity: $\rho \underline{v}$ = lin. mom.

advective mom. flux : $\rho \underline{v} \otimes \underline{v}$ (non. linear)

diffusive mom. flux : $-\underline{\underline{\sigma}}$

III Balance of angular momentum

$\underline{j} = \underline{x} \times \rho \underline{v} \qquad \frac{d}{dt}\int \underline{j}\, dV = \underline{\tau}_b + \underline{\tau}_s$

$\wedge$

$$\frac{d}{dt}\int_{\Omega_t} (\underline{x} \times \rho \underline{v})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b}\, dV_x$$

lhs

$$\frac{d}{dt}\int_{\Omega_t}\rho(\underline{x}\times\underline{v})\,dV_x = \int_{\Omega_t}\rho\frac{d}{dt}(\underline{x}\times\underline{v})\,dV_x$$

$$= \int_{\Omega_t}\rho(\underbrace{\dot{\underline{x}}\times\underline{v}}_{\underline{v}\times\underline{v}=\underline{0}}+\underline{x}\times\dot{\underline{v}})\,dV_x \qquad \dot{\underline{x}}=\underline{v}$$

$$= \int_{\Omega_t}\rho(\underline{x}\times\dot{\underline{v}})\,dV_x$$

rhs. subst. Cauchy stress

$$\int_{\Omega_t}\rho(\underline{x}\times\dot{\underline{v}})\,dV_x = \int_{\partial\Omega_t}\underline{x}\times\underline{\underline{\sigma}}\,\underline{n}\,dA + \int_{\Omega_t}\rho(\underline{x}\times\underline{b})\,dV_x$$

$$\int_{\Omega_t}\underline{x}\times(\rho\dot{\underline{v}}-\rho\underline{b})\,dV_x = \int_{\partial\Omega_t}\underline{x}\times\underline{\underline{\sigma}}\,\underline{n}\,dA_x$$

$$\rho\dot{\underline{v}}\boxed{-\nabla\cdot\underline{\underline{\sigma}}=\rho\underline{b}}$$

$$\int_{\Omega_t}\underline{x}\times(\nabla\cdot\underline{\underline{\sigma}})\,dV_x = \int_{\partial\Omega_t}\underline{x}\times\underline{\underline{\sigma}}\,\underline{n}\,dA_x$$

this is exactly the same statement as for the static case lecture 10 on Mechanical Equlbm $\Rightarrow$ ~~static~~

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \Rightarrow$ extends to transient case