Lecture 19: Continuum Thermodynamics

Logistics: - HW7 due Thursday

- Projects: - Alexandra, Kevin, Mbarak (Tu) ✓

  - Josh, Kaitlin, Huiwen, ChieChen (Wed)

  - Eric, Mikayla

Last time: - Local Eulerian Balance x

- mass: $\dot{\rho} + \rho \nabla_x \cdot \underline{v} = 0$ $\qquad \frac{D\rho}{Dt}$

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \underline{v}) = 0$$

- lin. mom.: $\rho \dot{\underline{v}} - \nabla_x \cdot \underline{\underline{\sigma}} = \rho \underline{b}$

$$\rho \frac{\partial \underline{v}}{\partial t} + \underbrace{(\underline{v} \cdot \nabla_x) \underline{v}}_{} - \nabla_x \cdot \underline{\underline{\sigma}} = \rho \underline{b} \qquad (\nabla_x \underline{v}) \underline{v}$$

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}$$

- ang. mom.: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Today: - Energy & entropy balance

- stress power

- internal dissipation density

- reversible processes

Balance of energy in local Eulerian Form

Net working in Eulerian Form

Power: $P = \underline{f} \cdot \underline{v}$ $\qquad \underline{f} = \int_\Omega \rho \underline{a} \, dV_x$

Newtons 2^{nd} law: $\underline{f} = m\underline{a} = m\dot{\underline{v}} \rightarrow \cancel{\underline{f} = \frac{d}{dt}(m\underline{v})} = m\dot{\underline{v}}$

Start taking dot product

$$(\rho\dot{\underline{v}})\cdot\underline{v} = \rho(\dot{\underline{v}}\cdot\underline{v}) = \rho\underline{v}\cdot\dot{\underline{v}} \qquad \rho\dot{\underline{v}} - \nabla_x\cdot\underline{\underline{\sigma}} = \rho\underline{b}$$

$$= (\nabla_x\cdot\underline{\underline{\sigma}})\cdot\underline{v} + \rho\underline{b}\cdot\underline{v} \qquad \rho\dot{\underline{v}} = \nabla_x\cdot\underline{\underline{\sigma}} + \rho\underline{b}$$

integrating

$$\int_{\Omega_t} \rho\underline{v}\cdot\dot{\underline{v}} \, dV_x = \int_{\Omega_t} (\nabla\cdot\underline{\underline{\sigma}})\cdot\underline{v} + \rho\underline{b}\cdot\underline{v} \, dV_x$$

identity : $\nabla\cdot(\underline{\underline{A}}^T\underline{c}) = \underbrace{(\nabla\cdot\underline{\underline{A}})\cdot\underline{c}} + \underline{\underline{A}} : \nabla\underline{c}$

$$(\nabla\cdot\underline{\underline{\sigma}})\cdot\underline{v} = \nabla_x\cdot(\underline{\underline{\sigma}}^T\underline{v}) - \underline{\underline{\sigma}} : \nabla_x\underline{v}$$

$$\int_{\Omega_t} \rho\underline{v}\cdot\dot{\underline{v}} \, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \nabla_x\underline{v} + \nabla_x\cdot(\underline{\underline{\sigma}}^T\underline{v}) + \rho\underline{b}\cdot\underline{v} \, dV_x$$

$$\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \qquad \underset{\substack{\uparrow \\ \text{sym}}}{\underline{\underline{\sigma}}} : \nabla_x\underline{v} = \underline{\underline{\sigma}} : \underbrace{\text{sym}(\nabla_x\underline{v})}$$

$$= \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}$$

here $\quad \dot{\underline{\underline{\varepsilon}}} = \underline{\underline{d}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^T\right)$ strain rate tensor

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} + \rho \underline{b} \cdot \underline{v} \, dV + \int_{\partial\Omega} \underline{\underline{\sigma}}\underline{v} \cdot \underline{n} \, dA_x$$

use transpose $\quad \underline{\underline{\sigma}}\underline{v} \cdot \underline{n} = \underline{v} \cdot \underline{\underline{\sigma}}^T \underline{n} = \underline{v} \cdot \underline{\underline{\sigma}}\underline{n} =$

$= \underline{v} \cdot \underline{t}$

$$\underbrace{\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x}_{\frac{d}{dt} K[\Omega_t]} = \int_{\Omega_t} -\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} \, dV_x + \underbrace{\overbrace{\int_{\Omega_t} \rho \underline{b} \cdot \underline{v} \, dV_x}^{\underline{f}_b \cdot \underline{v}} + \overbrace{\int_{\partial\Omega_t} \underline{t} \cdot \underline{v} \, dA_x}^{\underline{f}_s \cdot \underline{v}}}_{P[\Omega_t]}$$

Identify lhs as

$$\frac{d}{dt} K[\Omega_t] = \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \rho \underbrace{\underline{v} \cdot \underline{v}}_{|\underline{v}|^2} \, dV_x = \frac{1}{2} \int_{\Omega_t} \rho \frac{d}{dt}(\underline{v} \cdot \underline{v}) \, dV_x$$

$$\frac{d}{dt}(v_i v_i) = \dot{v}_i v_i + v_i \dot{v}_i = 2 v_i \dot{v}_i = 2 \underline{v} \cdot \dot{\underline{v}}$$

$$\frac{d}{dt} K[\Omega_t] = \int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV = \text{l.h.s}$$

$$\frac{d}{dt} K[\Omega_t] = \int_{\Omega_t} -\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} \, dV + P[\Omega_t]$$

by comparison $- \mathbb{W}[\Omega_t]$

with integral ~~co~~ balance law

$$\mathbb{W}[\Omega_t] = P[\Omega_t] - \frac{d}{dt} K[\Omega_t]$$

Net working on Ω_t

$$\boxed{\mathbb{W}[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} \, dV_x}$$

Quantity $\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}$ is called stress power associated with a motion. Corresponds to the rate of work done by internal forces (stresses) in a continuum body.

Local Eulerian form of First law

Integral form

$$\frac{d}{dt} U[\Omega_t] = Q[\Omega_t] + \mathbb{W}[\Omega_t]$$

where
$$U[\Omega_t] = \int_{\Omega_t} \rho u \, dV_x$$
$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} \underline{q} \cdot \underline{n} \, dA_x$$

$$\mathbb{W}[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}\, dV_x$$

Substituting

$$\frac{d}{dt}\int_{\Omega_t} \rho u \, dV_x = \int_{\Omega_t} \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}\, dV_x + \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} q \cdot \underline{n}\, dA_x$$

using "derivative with respect to mass" and div. Thm

$$\int_{\Omega_t} (\rho \dot{u} - \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} + \nabla_x \cdot q - \rho r)\, dV_x = 0$$

by arbitrariness of $\Omega_t \Rightarrow$ integrand is zero

$$\boxed{\rho \dot{u} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla \cdot q + \rho r}$$

local Eulerian form of energy balance

To write in conservative form

$$\rho \dot{u} = \rho \left(\frac{\partial u}{\partial t} + \nabla u \cdot \underline{v} \right) = \frac{\partial}{\partial t}(\rho u) - u \frac{\partial \rho}{\partial t} + \rho \nabla_x u \cdot \underline{v}$$

use mass cons. $\frac{\partial \rho}{\partial t} = -\nabla_x \cdot (\rho \underline{v})$

$$\rho \dot{u} = \frac{\partial}{\partial t}(\rho u) + \underbrace{u \nabla_x \cdot (\rho \underline{v}) + (\nabla_x u) \cdot (\rho \underline{v})}_{\nabla_x \cdot (\rho u \underline{v}) =}$$

$$\rho \dot{u} = \frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot (\rho u \underline{v})$$

substituting

Conservative local form of energy balance

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot (\rho u \underline{v} + \underline{q}) = \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} + \rho r}$$

$u \approx c_p T$

conserved quantity: ρu energy/volume

advective energy flux: $(\rho u)\,\underline{v}$

"diffusive energy flux: $\underline{q} = -k \nabla T$ (Fourier's law)

heating by dissipation: $\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}$
(shear heating

vol. heating rate: ρr

Local Eulerian Form of 2nd Law

$dS = \frac{dG}{T}$

Integral form of Clausius-Duhem θ = Temp.

$$\frac{d}{dt}\int_{\Omega} \rho s \, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\theta} dV_x - \int_{\partial\Omega_t} \frac{\underline{q} \cdot \underline{n}}{\theta} dA_x$$

apply div. Theorem + deriv. with density

$$\int_{\Omega_t} \rho \dot{s} \, dV \geq \int_{\Omega_t} \frac{\rho r}{\theta} - \nabla_x \cdot \left(\frac{\underline{q}}{\theta}\right) dV$$

$$\boxed{\rho \dot{s} \geq \frac{\rho r}{\theta} - \nabla_x \cdot \left(\frac{q}{\theta}\right)}$$ local eulerian form of CD inequality

multiplying by θ and expanding div.

$$\theta \rho \dot{s} \geq \rho r - \nabla_x \cdot q + \frac{q}{\theta} \cdot \nabla_x \theta$$

which can be written in terms of internal dissipation density $\delta = \theta \rho \dot{s} - (\rho r - \nabla_x \cdot q)$

$$\delta - \frac{q}{\theta} \cdot \nabla \theta \geq 0$$

δ is difference between local heating and local entropy increase

Note:

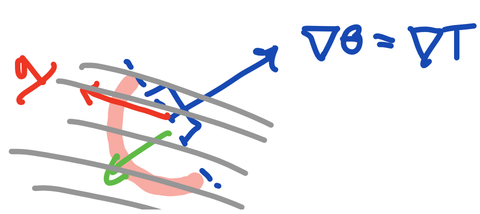
I) If $\nabla_x \theta = 0 \Rightarrow \delta \geq 0$

$\Rightarrow$ bodies with uniform θ have a non-neg. dissipation

II) If $\delta = 0$, i.e. reversible process

$$q \cdot \nabla_x \theta \leq 0$$

$$q \cdot \nabla T \leq 0$$



Thus heat flows down the temperature gradient

$$\underline{q} = -\underline{\underline{k}} \nabla T$$