Lecture 2: Tensor algebra (shit hits the fan)

Logistics: - HW 1 will be posted later today
due Sept 1 before class

- Let me know ahead of time if you need to hand in late (Mon)
- late HW -10 %
- Office hours: $\boxed{\begin{array}{ll} \text{Mon} & 2\text{-}3\ \text{pm} \\ \text{Wed} & 2\text{-}3\ \text{pm} \end{array}}$

(Tue 11 - noon) - Afzal → zoom

Last time: • Reviewed vectors

- scalar product: $\underline{a} \cdot \underline{b} = \alpha$
- vector product: $\underline{a} \times \underline{b} = \underline{c}$
- basis for vector space: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

orthonormal right handed basis ⇒ frame

• Index notation

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i = a_i \underline{e}_i$$

- twice repeated index → Dummy index

  → summation (Einstein notation)

- single indices → free index

  → group of equations

- Kronecker delta: $\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i=j \\ 0 & \end{cases}$

  → needed for scalar prod. in index not.

- Permutation symbol:

$$\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k = \begin{cases} 1 & \text{if even perm} \\ -1 & \text{if odd perm} \\ 0 & \text{repeated} \end{cases}$$

  → needed for cross product

Today:
- Index notation
  - Epsilon-Delta Identities
- Tensor algebra
  - second-order tensors
  - dyadic product
- Transpose, trace, scalar product

Frame identities

Summarize the relations between basis vectors

$\boxed{\underline{e}_j = \delta_{ij}\,\underline{e}_i}$ and $\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\,\underline{e}_k}$

consequence of orthonormal frame.

Epsilon-Delta identities

For any frame $\{\underline{e}_i\}$

$$\boxed{\begin{aligned} \epsilon_{pqs}\,\epsilon_{nrs} &= \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn} \\ \epsilon_{pqs}\,\epsilon_{rqs} &= 2\,\delta_{pr} \end{aligned}}$$

very useful in deriving all sorts of identities

Example: $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c} = \underline{d}$

$\underline{a} = a_q\,\underline{e}_q \qquad \underline{b} = b_i\,\underline{e}_i \qquad \underline{c} = c_j\,\underline{e}_j \qquad \underline{d} = d_p\,\underline{e}_p$

$$\underline{b} \times \underline{c} = (b_i\,\underline{e}_i) \times (c_j\,\underline{e}_j) = b_i\,c_j\,\underbrace{\underline{e}_i \times \underline{e}_j}_{\epsilon_{ijk}\,\underline{e}_k}$$

$$= \underbrace{\epsilon_{ijk}\,b_i\,c_j}\,\underline{e}_k$$

$$f_k\,\underline{e}_k = \underline{f}$$

$$\underline{a} \times \underline{f} = (a_q \underline{e}_q) \times (\epsilon_{ijk} b_i c_j \underline{e}_k)$$

$$= \epsilon_{ijk} a_q b_i c_j \underbrace{\underline{e}_q \times \underline{e}_k}_{\epsilon_{qkp} \underline{e}_p}$$

$$= \epsilon_{ijk} \epsilon_{qkp} a_q b_i c_j \underline{e}_p$$

$$(\epsilon_{qkp} = -\epsilon_{qpk} = \epsilon_{pqk})$$

$$= \boxed{\epsilon_{ijk} \epsilon_{pqk} a_q b_i c_j \underline{e}_p = \underline{a} \times (\underline{b} \times \underline{c})}$$

index notation expression for

$a \times (b \times c)$

Apply $\varepsilon\delta$ identity

$$= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) a_q b_i c_j \underline{e}_p$$

$$= \delta_{ip} \delta_{jq} a_q b_i c_j \underline{e}_p - \delta_{iq} \delta_{jp} a_q b_i c_j \underline{e}_p$$

use transfer property of δ

$$= a_j b_i c_j \underline{e}_i - a_i b_i c_j \underline{e}_j$$

$$= (a_j c_j)\, b_i \underline{e}_i - (a_i b_i)\, c_j \underline{e}_j$$

$$(\underline{a}\cdot\underline{c})\,\underline{b} \quad - \quad (\underline{a}\cdot\underline{b})\,\underline{c}$$

$$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$$

HW 1: $(\underline{a}\times\underline{b})\times\underline{c} = \cancel{a} \ \ldots$

Second-order tensors

Linear operator: $\underline{v} = \underline{\underline{A}}\,\underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

linearity requires

1) $\underline{\underline{A}}(\underline{u}+\underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$ for all $\underline{u},\underline{v} \in \mathcal{V}$

2) $\underline{\underline{A}}(\alpha\,\underline{v}) = \alpha\,\underline{\underline{A}}\,\underline{v}$ for all $\underline{v} \in \mathcal{V}$, $\alpha \in \mathbb{R}$

Example: $\underline{\underline{A}}$ maps every $\underline{v} \in \mathcal{V}$ into $\underline{n} \neq \underline{0} \in \mathcal{V}$

Is $\underline{\underline{A}}$ a tensor?

Consider $\underline{u}, \underline{v}, \underline{w}$

$$\underline{w} = \underline{u} + \underline{v}$$

use first requirement:

$$\underline{\underline{A}}(\underbrace{\underline{u}+\underline{v}}_{\underline{w}}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underbrace{\underline{\underline{A}}\,\underline{w}} = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{n} \neq \underline{n} + \underline{n}$$

$$\underline{n} \neq 2\,\underline{n}$$

$\Rightarrow$ $\underline{\underline{A}}$ is not a tensor

Tensor algebra

1) scalar multiplication: $(\alpha\,\underline{\underline{A}})\,\underline{v} = \underline{\underline{A}}(\alpha\,\underline{v})$
2) tensor sum: $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v}$
3) tensor product: $(\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{v})$

for all $\underline{v} \in \mathcal{V}$

Note: tensor scalar product introduced later

The set of all 2^{nd}-order tensors $\mathcal{V}^2$ is a vector space, i.e., closed under these 3 operations

1) $\alpha \underline{\underline{A}} \in \mathcal{V}^2$ for all $\underline{\underline{A}} \in \mathcal{V}^2$

2) $\underline{\underline{A}} + \underline{\underline{B}} \in \mathcal{V}^2$ for all $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$

3) $\underline{\underline{A}}\,\underline{\underline{B}} \in \mathcal{V}^2$ " "

Q: what is the basis for $\mathcal{V}^2$?

Two tensors $\underline{\underline{A}}$ & $\underline{\underline{B}}$ are equal

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \quad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ " "

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ " "

Representation of tensor

In frame $\{\underline{e}_i\}$ a 2nd-order tensor $\underline{\underline{S}}$ is represented by nine ~~[illegible]~~ components

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Matrix representation of $\underset{=}{S}$ in $\{\underline{e}_i\}$

$$[\underset{=}{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$[\underset{=}{S}]_{ij} = S_{ij}$$

Consider $\underline{v} = \underset{=}{S}\,\underline{u}$ in index notation

$$\underline{v} = v_k\,\underline{e}_k \qquad \underline{u} = u_j\,\underline{e}_j$$

$$v_k\,\underline{e}_k = \underset{=}{S}\,(u_j\,\underline{e}_j) = \underset{=}{S}\,\underline{e}_j\,u_j$$

$$v_k\,\underbrace{\underline{e}_i\cdot\underline{e}_k} = \underbrace{\underline{e}_i\cdot\underset{=}{S}\,\underline{e}_j}\,u_j$$

$$\delta_{ik}\,v_k = S_{ij}\,u_j$$

$$\boxed{v_i = S_{ij}\,u_j}$$

j = dummy

i = free

Dyadic Product

The dyadic product between two vectors $\underline{a}$ and $\underline{b}$ is the sec. order tensor $\underline{a} \otimes \underline{b}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\, \underline{v} = (\underline{b} \cdot \underline{v})\, \underline{a}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

This has form $\underline{\underline{A}}\, \underline{v} = \alpha\, \underline{a}$

$$A_{ij} v_j = \alpha\, a_i$$

$$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$$

$$A_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij} v_j = b_j v_j a_i$$

$$= a_i b_j v_j$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$$

so that

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\, \underline{b}^T}$$

If every vector is column vector

$$\underline{a} \cdot \underline{b} \;\; \rightarrow \;\; \underline{a}^T \cdot \underline{b} = \underset{1\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \underset{1\times 1}{a_1 b_1 + a_2 b_2 + a_3 b_3}$$

$$\underline{a}\,\underline{b}^T = \underset{3\times 1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots \\ & \ddots & \\ & & \ddots \end{bmatrix}}$$

some naughty people $\underline{a}\,\underline{b}^T = \underline{a}\,\underline{b}$

Linearity of dyadic product

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha (\underline{a} \otimes \underline{b}) \underline{v} + \beta (\underline{a} \otimes \underline{b}) \underline{w}}$$

The product of two dyadic products

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}}$$ HW 2

$\Rightarrow$ needed for tensor product in index notation

Basis for $\mathcal{V}^2$ (vector space of 2nd-ord. tensors)

Given any frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for $\mathcal{V}^2$. Any 2nd-order tensor $\underline{\underline{S}}$ can be written as linear combination

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j}$$

$$= S_{11}\, \underline{e}_1 \otimes \underline{e}_1 + S_{12}\, \underline{e}_1 \otimes \underline{e}_2 + \ldots$$

this is analogous $\quad \underline{v} = v_i\, \underline{e}_i$

$$\underline{e}_1 \otimes \underline{e}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{e}_2 \otimes \underline{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider $\quad \underline{v} = \underline{\underline{S}}\, \underline{u} \quad$ with $\underline{v} = v_i\, \underline{e}_i \quad \underline{u} = u_k\, \underline{e}_k$

$$v_i\, \underline{e}_i = (S_{ij}\, \underline{e}_i \otimes \underline{e}_j)(u_k\, \underline{e}_k) \qquad \underline{(\underline{a} \otimes \underline{b})\, \underline{v} = (\underline{b} \cdot \underline{v})\, \underline{a}}$$

$$= S_{ij}\, u_k\, (\underline{e}_i \otimes \underline{e}_j)\, \underline{e}_k$$

$$\underbrace{(\underline{e}_j \cdot \underline{e}_k)}\, \underline{e}_i$$

$$\delta_{jk}\, \underline{e}_i$$

$$= S_{ij}\, n_k\, \delta_{jk}\, \underline{e}_i$$

$$v_i\, \underline{e}_i = \underline{S_{ij}\, u_j}\, \underline{e}_i \qquad v_i = S_{ij}\, u_j$$