Lecture 20: Lagrangian local balance laws

Logistics: HW 7 is due

→ most people have projects   Nye

happy to discuss further during

office hours next week

Last time: - Continuum thermodynamics

1st law: $\rho \dot{u} = -\nabla \cdot \underset{\sim}{q} + \underset{\approx}{\sigma} : \dot{\underset{\approx}{\varepsilon}} + \rho r$

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \underset{\sim}{v} + \underset{\sim}{q}) = \underset{\approx}{\sigma} : \dot{\underset{\approx}{\varepsilon}} + \rho r$$

2nd law: $\rho \dot{s} \geq \frac{\rho r}{\theta} - \nabla_x \cdot \left(\frac{\underset{\sim}{q}}{\theta}\right)$ Claus. Duhem.

⇒ direction of heat conduction

Today: Eulerian ($\underset{\sim}{x}$) → Lagrangian ($\underset{\sim}{X}$)

Summary of Eulerian Balance Laws

Equations

mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$ 1

lin. mom: $\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}$ 3

ang. mom: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \quad \sigma_{12} = \sigma_{21} \quad \sigma_{13} = \sigma_{31} \quad \sigma_{23} = \sigma_{32}$ 3

energy: $\frac{\partial}{\partial t}(\rho u) + \nabla \cdot [\rho u \underline{v} + \underline{q}] = \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} + \rho r$ 1

Kinematic: $v_m = \frac{\partial \varphi}{\partial t}$ + 3

11

Unknown fields:

$\underline{\varphi}$	$\underline{v}$	ρ	$\underline{\underline{\sigma}}$	θ	$\underline{q}$	u	
3	3	1	9	1	3	1	= 21

We have 21 unknowns but only 11 equations

$\Rightarrow$ under constrained

Need additional constitutive equations

Remarks:

1) Eulerian formulation the balance laws are independent of $\underline{\varphi}$!

Motion is only needed to determine the shape of the domain.

If B_t is known 8 equs & $\underbrace{18}$ unknowns

Need constitutive relations that relate $\underline{\underline{\sigma}}$, $\underline{q}$ and u to ρ, $\underline{v}$ and θ.

Examples: $\underline{q} = -K \nabla \theta$ or $u = \rho c_p(\theta)\,\theta$

2) Neglecting thermal effects then $\overset{3}{\underline{q}}$, $\overset{1}{u}$, $\overset{1}{\theta}$ dissapear and we have 7 equs but 12 unknowns

Need constitutive relations that relate $\underline{\underline{\sigma}}$ to ρ & $\underline{v}$

Example: Ideal isothermal fluid

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} \qquad \rho = \rho(p) \qquad \rho = \rho_0\left(1 + \beta(p - p_0)\right)$$

mass: $\frac{\partial}{\partial t}\rho(p) + \nabla\cdot\left(\rho(p)\,\underline{v}\right) = 0$

lin. mom.: $\frac{\partial}{\partial t}\left(\rho(\cancel{p})\underline{v}\right) + \nabla\cdot\left(\rho(\cancel{p})\underline{v}\otimes\underline{v} + p\underline{\underline{I}}\right) = \rho(\cancel{p})\underline{b}$

4 unknowns $\underline{v}$ and p and 4 equs

$\Rightarrow$ closed system

Example: Incompressible Navier Stokes

$\rho = \rho_0 \quad \Rightarrow \quad \nabla\cdot\underline{v} = 0$ "continuity equ."

$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu\left(\nabla\underline{v} + \nabla\underline{v}^T\right)$

substitute into lin. mom.

$$\frac{\partial}{\partial t}(\rho_0\underline{v}) + \nabla\cdot\left[\rho_0\underline{v}\otimes\underline{v} + p\underline{\underline{I}} - \mu\left(\nabla\underline{v} + \nabla\underline{v}^T\right)\right] = \rho_0\underline{b}$$

if $\mu =$ const.

$$\rho_0\dot{\underline{v}} = \mu\nabla^2\underline{v} - \nabla p + \rho_0\underline{b}$$

$$\dot{\underline{v}} = \nu\nabla^2\underline{v} - \frac{\nabla p}{\rho_0} + b$$

dynamic viscosity

$\nu = \frac{\mu}{\rho_0}$

kinematic viscosity

Lagrangian balance laws

in terms of $\underline{X}$

1) By change of variables from Eulerian $\underline{X} = \underline{\varphi}^{-1}(\underline{x}, t)$

2) straight from integral balance laws

$\rightarrow$ ~~RH~~ 2

I Balance of mass

We already did this

$$\boxed{\rho_m(\underline{X}, t)\, J(\underline{X}, t) = \rho_0(\underline{X})} \qquad J = \det(\underline{\underline{F}})$$

mass density is known in Lag. formulation

II Balance of Linear Momentum

Integral balance law

$$\frac{d}{dt}\, \underline{L}[\Omega_t] = \underline{r}[\Omega_t]$$

where

$$\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x}, t)\, \underline{v}(\underline{x}, t)\, dV_x$$

$$\underline{r}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x}, t)\, \underline{b}(\underline{x}, t)\, dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x$$

change variable from $\underline{x} \to \underline{X}$

$$\underline{L}[\Omega_t] = \int_{\Omega} \rho_m(\underline{X},t)\, \underbrace{\underline{v}_m(\underline{X},t)}_{\underline{V}(\underline{X},t) = \dot{\underline{\varphi}}(\underline{X},t)}\, J(\underline{X},t)\, dV_X \qquad \rho_0(\underline{X}) = \rho_m J$$

$$= \int_{\Omega} \rho_0(\underline{X})\, \dot{\underline{\varphi}}(\underline{X},t)\, dV_X$$

For change of variables on rhs $\underline{\Gamma}_S$ we need Nansons formula: $\underline{n}\, dA_x = J \underline{\underline{F}}^{-T} \underline{N}\, dA_X$

$$\underline{\Gamma}[\Omega_t] = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho\, \underline{b}\, dV_x$$

$$= \int_{\partial\Omega} \underline{\underline{\sigma}}_m J \underline{\underline{F}}^{-T} \underline{N}\, dA_X + \int_{\Omega} \underbrace{\rho_m\, b_m J}_{\rho_0}\, dV_X$$

To simplify notation we introduce the tensor

$$\boxed{\underline{\underline{P}}(\underline{X},t) = J(\underline{X},t)\, \underline{\underline{\sigma}}_m(\underline{X},t)\, \underline{\underline{F}}(\underline{X},t)^{-T}}$$

first Piola-Kirchhoff stress tensor

$$\underline{\Gamma}[\Omega_t] = \int_{\partial\Omega} \underline{\underline{P}}\, \underline{N}\, dA_X + \int_{\Omega} \rho_0\, \underline{b}_m\, dV_X$$

substitute into ~~the~~ Integral balance law:

$$\frac{d}{dt}\int_{\Omega} \rho_0 \dot{\underline{\varphi}}\, dV_X = \int_{\partial\Omega} \underline{\underline{P}}\,\underline{N}\, dA_X + \int_{\Omega} \rho_0 \underline{b}_m\, dV_X$$

since $\Omega \neq \Omega(t)$ and $\rho_0 \neq \rho_0(t)$

$$\int_{\Omega} \rho_0 \ddot{\underline{\varphi}}\, dV_X = \int_{\Omega} \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m\, dV_X$$

by arbitrary ... of Ω

$$\boxed{\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m}$$

local Lagrangian form of lin. mom. bal.

Note: $\underline{\underline{P}}$ is natural stress tensor in the material description because it relates the traction on a surface to its normal.

spatial: $\underline{t}(\underline{x},t) = \underline{\underline{s}}(\underline{x},t)\,\underline{n}$

material: $\underline{T}(\underline{X},t) = \underline{\underline{P}}(\underline{X},t)\,\underline{N}$

$\underline{T}$ is the (nominal) Piola-Kirchhoff traction vector

$\underline{t}$ is the (true) Cauchy traction vector

The resultant force

$$d\underline{f} = \underline{t}\, dA_x = \underline{T}\, dA_X$$

$$\underline{t} = \frac{f}{A}$$

$\Rightarrow \quad \underline{T} \parallel \underline{t}$

III) Balance of angular momentum

Cauchy stress $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

by definition: $\quad \underline{\underline{P}} = J\, \underline{\underline{\sigma}}_m\, \underline{\underline{F}}^{-T}$

$$\underline{\underline{\sigma}}_m = \frac{1}{J}\, \underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{\sigma}}_m^T = \left(\frac{1}{J}\, \underline{\underline{P}}\, \underline{\underline{F}}^T\right)^T = \frac{1}{J}\, \underline{\underline{F}}\, \underline{\underline{P}}^T$$

$$\Rightarrow \quad \boxed{\underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{F}}\, \underline{\underline{P}}^T}$$

$\Rightarrow \underline{\underline{P}} \neq \underline{\underline{P}}^T \Rightarrow \underline{\underline{P}}$ has 9 independent entries

Motivates definition of second Piola-Kirchhoff stress tensor $\quad \boxed{\underline{\underline{\Sigma}} = \underline{\underline{F}}\, \underline{\underline{P}}^T} \qquad \underline{\underline{\Sigma}}^T = \underline{\underline{\Sigma}}$

Networking in Lagrangian form

$$\mathbb{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} K[\Omega_t]$$

From integral balance laws:

$$K[\Omega_t] = \frac{1}{2}\int_{\Omega_t} \rho |\underline{v}|^2 \, dV_x$$

$$J\underline{\underline{F}}^{-T}\underline{N}\, dA_X$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega_t} \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma_n}}\, \underline{v_n} \cdot \overbrace{\underline{n}\, dA_x}$$

where we have use $\underline{v} \cdot \underline{\underline{\sigma}}\, n = \underline{\underline{\sigma}}^T \underline{v} \cdot n = \underline{\underline{\sigma}}\, \underline{v} \cdot \underline{n}$

Change in variables

$$\boxed{K[\Omega_t] = \int_{\Omega} \frac{1}{2} \rho_0 |\dot{\underline{\varphi}}|^2 \, dV_X}$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega} \rho_0 \underline{b}_m \cdot \underline{v}_m \, dV_X + \int_{\partial\Omega} J\, \underline{\underline{\sigma}}_m \underline{v}_m \cdot \underline{\underline{F}}^{-T}\underline{N}\, dA_X$$

$$= \int_{\Omega} \rho_0 \underline{b}_m \cdot \underbrace{\underline{v}_m}_{\dot{\underline{\varphi}}} \, dV_X + \int_{\partial\Omega} J\, \underbrace{\underline{v}_m}_{\dot{\underline{\varphi}}} \cdot \underbrace{\underline{\underline{\sigma}}_m \underline{\underline{F}}^{-T}}_{\underline{\underline{P}}} \underline{N}\, dA_X$$

$$= \int_{\Omega} \rho_0 \underline{b}_m \cdot \dot{\underline{\varphi}} \, dV_X + \int_{\partial\Omega} \underbrace{\dot{\underline{\varphi}} \cdot \underline{\underline{P}}\, \underline{N}}\, dA_X$$

$$\underline{\underline{P}}^T \dot{\underline{\varphi}} \cdot \underline{N}$$

$$\nabla_X \cdot (P^T \dot{\underline{\varphi}})$$

$$=$$

$$= \int_{\Omega} \rho_0 b_m \cdot \dot{\underline{\varphi}} + \nabla_X \cdot (\underline{\underline{P}}^T \dot{\underline{\varphi}})\, dA_X$$

use identity $\nabla_X \cdot (\underline{\underline{P}}^T \dot{\underline{\varphi}}) = (\nabla_X \cdot \underline{\underline{P}}) \cdot \dot{\underline{\varphi}} + \underline{\underline{P}} : \underbrace{\nabla_X \dot{\underline{\varphi}}}_{\dot{\underline{\underline{F}}}}$

power in Lagrangian form.

$$\boxed{\mathcal{P}[\Omega_t] = \int_\Omega [\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m] \cdot \dot{\underline{\varphi}} + \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_X}$$

Rate of change of K

$$\frac{d}{dt} K[\Omega_t] = \frac{d}{dt} \int_\Omega \frac{1}{2} \rho_0 |\dot{\underline{\varphi}}|^2 dV_X$$

$$= \int_\Omega \frac{1}{2} \rho_0 \frac{d}{dt} |\dot{\underline{\varphi}}|^2 dV_X = \int_\Omega \dot{\underline{\varphi}} \cdot (\rho_0 \ddot{\underline{\varphi}}) \, dV_X$$

$$\frac{d}{dt} (\dot{\underline{\varphi}} \cdot \dot{\underline{\varphi}}) = \dot{\underline{\varphi}} \ddot{\underline{\varphi}} + \ddot{\underline{\varphi}} \dot{\underline{\varphi}} = 2 \dot{\underline{\varphi}} \ddot{\underline{\varphi}}$$

lin. mom. balance: $\rho \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m$

$$\frac{d}{dt} K[\Omega_t] = \int_\Omega (\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m) \cdot \dot{\underline{\varphi}} \, dV$$

$$= \mathcal{P}[\Omega_t] - \int_\Omega \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_X$$

Lagrangian def. of networking:

$$\boxed{W[\Omega_t] = \int_\Omega \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_X}$$

$\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}$, $\nabla \underline{v}^T$

$\dot{\underline{\underline{F}}} = (\nabla \underline{\varphi})^{\cdot}$

$= \nabla \dot{\underline{\varphi}}$

$\dot{\underline{v}}$

analogous to Eulerian case

$\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}$ power per unit volume of B_t

$\underline{\underline{P}} : \dot{\underline{\underline{F}}}$ power per unit volume of B

$\Rightarrow$ see notes $\quad \underline{Q} = J \underline{\underline{F}}^{-1} q_m$ material heat flux

$$\boxed{\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R}$$ local Lagrangian energy balance

$R = r_m \qquad U = u_m$

skip

<u>Lagrangian balance laws</u>

Kinematic :	$\underline{V} = \dot{\underline{\varphi}}$	3
lin. mom.:	$\rho_0 \dot{\underline{V}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 b_m$	3
ang mom.:	$\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$	3
energy :	$\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R$	1
		10

Lagrangian fields:

$\underline{\varphi}$	$\underline{v}$	$\underline{\underline{P}}$	Θ	$\underline{Q}$	u	
3	3	9	1	3	1	$= 20$

$\Rightarrow$ 20 unknown & 10 equs.

Notes:

1) In many situations $\underline{V}$ is not needed

$\Rightarrow$ 17 ~~equs~~ unknowns & 7 equs

2) Constitutive relations that relate

$\underline{\underline{\Sigma}}$ to φ

$\underline{Q}$, $\underline{u}$ to Θ

3) Isothermal system $\quad \underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\varphi})$