Lecture 21: Intro to constitutive theory

Logistics: - HW8 due Thu.

Last time: Lagrangian balance laws

mass: $\rho_m J = \rho_0$

Paths

lin. mom.: $\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m$

$\underline{\underline{\ell}} = \nabla_x \underline{v}$

ang. mom.: $\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T = \Sigma$

$\underline{\underline{\ell}} = \underline{\underline{d}} + \underline{\underline{w}}$ $\neq \dot{\underline{\underline{e}}}$

energy: $\rho_0 \dot{U} = \underset{\underline{\underline{\sigma}}}{\underline{\underline{P}}} : \underset{=\nabla_x v}{\dot{\underline{\underline{F}}}} - \nabla_X \cdot \underset{\underline{q}}{\underline{Q}} + \rho_0 \underset{r}{R}$

$\underline{\underline{d}} = \text{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$

1st Piola-Kirchhoff stress: $\underline{\underline{P}} = J\,\underline{\underline{\sigma}}_m\,\underline{\underline{F}}^{-T}$

2nd Piola-Kirchhoff stress: $\underline{\underline{\Sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T$

$\underline{\underline{w}} = \text{skew}(\underline{\underline{\ell}})$

Piola-Kirchhoff traction: $\underline{T} = \underline{\underline{P}}\,\underline{N}$

$\underline{\underline{d}} \neq \frac{d}{dt}\underline{\underline{\varepsilon}}$

Today: 4th order tensors

Constitutive theory

Constitutive Theory

Common constitutive laws:

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta(\nabla\underline{v} + \nabla\underline{v}^T)$

$p = -\frac{1}{3}tr(\underline{\underline{\sigma}})$ η = viscosity $\underline{v}$ = velocity

Linear elastic solid: $\underline{\underline{\sigma}} = \lambda \nabla\cdot\underline{u}\,\underline{\underline{I}} + \mu(\nabla\underline{u} + \nabla\underline{u}^T)$

λ, μ = Lamé param. $\underline{u}$ = displacement

Both derive from same functional form:

$$\boxed{\underline{\underline{\sigma}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\, tr(\underline{\underline{A}}) + 2\mu\, sym(\underline{\underline{A}})}$$

$\uparrow$ 4^{th} order tensor

linear isotropic constitutive law

Newtonian fluid: $\underline{\underline{A}} = \nabla_x \underline{v}$

Linear elastic solid: $\underline{\underline{A}} = \nabla_x \underline{u}$

remember $\nabla\cdot\underline{a} = tr(\nabla\underline{a})$

$\Rightarrow$ works directly for lin. elastic solid

for Newt. fluid it's more complicated due to incompressibility

Why this form?

Fourth-order tensors

So far "tensor" meant second-order tensor

$$\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

By a fourth-order tensor $\mathbb{C}$ we mean a mapping $\mathbb{C}: \mathcal{V}^2 \to \mathcal{V}^2$ which is linear, i.e.:

1) $\mathbb{C}(\underline{\underline{T}} + \underline{\underline{S}}) = \mathbb{C}\underline{\underline{T}} + \mathbb{C}\underline{\underline{S}}$ for all $\underline{\underline{T}}, \underline{\underline{S}} \in \mathcal{V}^2$
2) $\mathbb{C}(\alpha \underline{\underline{S}}) = \alpha\, \mathbb{C}\underline{\underline{S}}$ for all $\underline{\underline{S}} \in \mathcal{V}^2$, $\alpha \in \mathbb{R}$

The set of 4^{th} order tensors is denote $\mathcal{V}^4$

Zero tensor: $\mathbb{O}\underline{\underline{T}} = \underline{\underline{0}}$ for all $\underline{\underline{T}} \in \mathcal{V}^2$

Identity tensor: $\mathbb{I}\underline{\underline{T}} = \underline{\underline{T}}$ " " "

Simple example:

$\mathbb{C}\underline{\underline{T}} = \underline{\underline{A}}\,\underline{\underline{T}}$ defines 4^{th} order tensor

$$\mathbb{C}(\alpha \underline{\underline{S}} + \beta \underline{\underline{T}}) = \underline{\underline{A}}(\alpha \underline{\underline{S}} + \beta \underline{\underline{T}}) = \alpha \underline{\underline{A}}\,\underline{\underline{S}} + \beta \underline{\underline{A}}\,\underline{\underline{T}}$$

$$= \alpha\, \mathbb{C}\underline{\underline{S}} + \beta\, \mathbb{C}\underline{\underline{T}} \quad \checkmark$$

Forth-order tensor algebra

sum: $(\mathbb{C} + \mathbb{D})\,\underline{\underline{T}} = \mathbb{C}\underline{\underline{T}} + \mathbb{D}\underline{\underline{T}}$ for all $\underline{\underline{T}} \in \mathcal{V}^2$

prod: $(\mathbb{C}\,\mathbb{D})\,\underline{\underline{T}} = \mathbb{C}\,(\mathbb{D}\underline{\underline{T}})$ " " "

Representation of 4^{th} order tensor

The 81 components of $\mathbb{C}$ in frame $\{\underline{e}_i\}$ are

$$\boxed{\mathbb{C}_{ijkl} = \underline{e}_i \cdot \mathbb{C}\,(\underline{e}_k \otimes \underline{e}_l)\,\underline{e}_j}$$

$\mathbb{C}\,(\underline{e}_k \otimes \underline{e}_l) = \underline{\underline{S}}_{kl}$ is second order tensor

$$\mathbb{C}_{ijkl} = \underline{e}_i \cdot \underline{\underline{S}}_{kl}\,\underline{e}_j$$

Mapping between two second-order tensors

$\underline{\underline{U}} = U_{ij}\,\underline{e}_i \otimes \underline{e}_j$ and $\underline{\underline{T}} = T_{kl}\,\underline{e}_k \otimes \underline{e}_l$

What is $\underline{\underline{U}} = \mathbb{C}\,\underline{\underline{T}}$

$$U_{ij} = \underline{e}_i \cdot \underline{\underline{U}}\,\underline{e}_j = \underline{e}_i \cdot \mathbb{C}\underline{\underline{T}}\,\underline{e}_j = \underline{e}_i \cdot \mathbb{C}\,T_{kl}(\underline{e}_k \otimes \underline{e}_l)\,\underline{e}_j$$

$$= \underbrace{\underline{e}_i \cdot \mathbb{C}\,(\underline{e}_k \otimes \underline{e}_l)\,\underline{e}_j}_{\mathbb{C}_{ijkl}}\,T_{kl}$$

$$\boxed{U_{ij} = \mathbb{C}_{ijkl} T_{kl}}$$

Components of $\mathbb{C}$ are coefficients in linear mapping from T_{kl} to U_{ij}

Example: $\mathbb{C}\underline{\underline{T}} = \underline{\underline{A}}\,\underline{\underline{T}}$

$$\mathbb{C}_{ijkl} = \underline{e}_i \cdot \underline{\underline{A}}\,(\underline{e}_k \otimes \underline{e}_l)\,\underline{e}_j$$

$$= \underline{e}_i \cdot \underline{\underline{A}}\,(\underline{e}_l \cdot \underline{e}_j)\,\underline{e}_k = \underline{e}_i \cdot \underline{\underline{A}}\,\delta_{lj}\,\underline{e}_k$$

$$= \underbrace{\underline{e}_i \cdot \underline{\underline{A}}\,\underline{e}_k}_{A_{ik}}\,\delta_{lj} = A_{ik}\,\delta_{lj}$$

$$\Rightarrow \quad \mathbb{C}_{ijkl} = A_{ik}\,\delta_{lj}$$

Fourth-order dyadic products

The dyadic product of 4 vectors $\underline{a}, \underline{b}, \underline{c}$ & $\underline{d}$ is the fourth-order tensor $\underline{a}\otimes\underline{b}\otimes\underline{c}\otimes\underline{d}$ defined by

$$\boxed{(\underline{a}\otimes\underline{b}\otimes\underline{c}\otimes\underline{d})\,\underline{\underline{T}} = (\underline{c}\cdot\underline{\underline{T}}\,\underline{d})\,\underline{a}\otimes\underline{b}}$$

analogous to: $(\underline{a}\otimes\underline{b})\,\underline{c} = (\underline{b}\cdot\underline{c})\,\underline{a}$

Given some $\{\underline{e}_i\}$ the set of 81 dyadic products $\{\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\}$ forms a basis for $\mathcal{V}^4$.

$$\Rightarrow \quad \boxed{\mathbb{C} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l}$$

$$C_{ijkl} = \underline{e}_i \cdot \mathbb{C}(\underline{e}_k \otimes \underline{e}_l)\, \underline{e}_j$$

analogous: $\underline{\underline{A}} = A_{ij}(\underline{e}_i \otimes \underline{e}_j)$

$$A_{ij} = \underline{e}_i \cdot \underline{\underline{A}}\, \underline{e}_j$$

This gives correct expression for $\underline{\underline{U}} = \mathbb{C}\,\underline{\underline{T}}$

$$\underline{\underline{U}} = \mathbb{C}\,\underline{\underline{T}} = C_{ijkl}(\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l)\, \underline{\underline{T}}$$

$$= C_{ijkl}\,\underbrace{(\underline{e}_k \cdot \underline{\underline{T}}\,\underline{e}_l)}_{T_{kl}}\,(\underline{e}_i \otimes \underline{e}_j)$$

$$= \underbrace{C_{ijkl}\, T_{kl}}_{U_{ij}}\,(\underline{e}_i \otimes \underline{e}_j)$$

$$\underline{\underline{U}} = U_{ij}\,(\underline{e}_i \otimes \underline{e}_j)$$

Symmetries of 4^{th} order tensors

For 2^{nd}-order tensors: $\underline{A} = \underline{A}^T \quad A_{ij} = A_{ji}$

$\mathbb{C}$ has a major symmetry if

$$\boxed{\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathbb{C}\underline{\underline{A}} : \underline{\underline{B}}} \quad \text{for } \underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$$

In components major symmetry implies

$$\boxed{\mathbb{C}_{ijkl} = \mathbb{C}_{klij}}$$

$\mathbb{C}$ has a right minor symmetry if

$$\boxed{\begin{gathered}\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \underline{\underline{A}} : \mathbb{C}\,\mathrm{sym}(\underline{\underline{B}}) \\ \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}\end{gathered}}$$

$\mathbb{C}$ has a left minor symmetry if

$$\boxed{\begin{gathered}\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathrm{sym}(\underline{\underline{A}}) : \mathbb{C}\underline{\underline{B}} \\ \mathbb{C}_{ijkl} = \mathbb{C}_{jikl}\end{gathered}}$$

Major sym: $\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathbb{C}\underline{\underline{A}} : \underline{\underline{B}}$

$$\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = A_{ij}\,\mathbb{C}_{ijkl}B_{kl} = \underline{\mathbb{C}_{ijkl}}\,A_{ij}\,B_{kl}$$

$$\mathbb{C}\underline{\underline{A}} : \underline{\underline{B}} = \mathbb{C}_{ijkl}A_{kl}\,B_{ij} = \mathbb{C}_{ijkl}A_{kl}B_{ij} = \mathbb{C}_{\underline{klij}}A_{ij}B_{kl}$$

only same if $\mathbb{C}$

$i \to k$
$j \to l$
$k \to i$
$l \to j$

$\Rightarrow$ How to represent
4^{th} tensor as a matrix
Voight notation.