Lecture 22: Objectivity and Material constraints

Logistics: HW8 due

HW9 will be posted

Projects due Dec 6^{th}

Last time: • Constitutive laws

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\underline{\underline{A}} = \lambda \operatorname{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu \operatorname{sym}(\underline{\underline{A}})}$$

$A \rightarrow \nabla \underline{v}$ or $\nabla \underline{u}$

• What is a fourth-order tensor?

4-th order dyadic product

$$\boxed{(\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d})\underline{\underline{T}} = (\underline{c} \cdot \underline{\underline{T}}\underline{d})\, \underline{a} \otimes \underline{b}}$$

4-th order tensor

$$\mathbb{C} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\mathbb{C}_{ijkl} = \underline{e}_i \cdot \mathbb{C}(\underline{e}_k \otimes \underline{e}_l)\underline{e}_j$$

• $\boxed{\underline{\underline{U}} = \mathbb{C}\underline{\underline{T}}} \Rightarrow \boxed{U_{ij} = \mathbb{C}_{ijkl} T_{kl}}$

• Major & minor symmetries

Today: Objectivity, Representation Thm, Constraints

Change of observer

Lecture 6 we discussed change in basis

$\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ $\quad \underline{v} = \underline{\underline{Q}}\,\underline{v}' \qquad \underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$

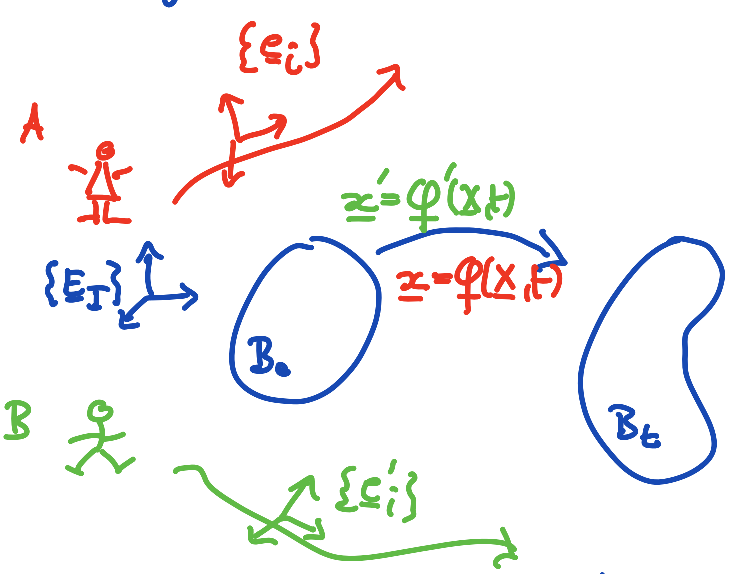
where $\underline{\underline{Q}}$ is a rotation (change in basis tensor)

1) orthonormal: $\underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T \Rightarrow \underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}$

2) $\det(\underline{\underline{Q}}) = 1$

Change in basis is <u>passive</u> change in frame.

Change in observer (active change in frame)



Material frame is common but spatial frame is different.

(Note: assume same clock)

Change in observer cannot induce a deformation $\Rightarrow$ φ and φ' must be related by a superposed rigid motion.

$$\underline{x}' = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t) \qquad \underline{x} = \varphi(\underline{X},t)$$

$$\boxed{\underline{x}' = \underline{\underline{Q}}(t)\,\varphi(\underline{X},t) + \underline{c}(t)} \quad \text{Eulerian transformation.}$$

$\underline{\underline{Q}}$ = rotation $\qquad \underline{c}$ = translation.

Objective description of forces and deformations cannot depend on the observer.

Effect of superposed rigid motion of kinematic quantities

$$\underline{x} = \varphi(\underline{X},t) \qquad \nabla_X \varphi = \underline{\underline{F}}$$

$$\underline{x}' = \varphi'(\underline{X},t) = \underline{\underline{Q}}\,\varphi(\underline{X},t) + \underline{c}(t)$$

$$\nabla\varphi' = \underline{\underline{Q}}\,\nabla\varphi = \underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}' \qquad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$$

Right Cauchy-Green strain tensor: $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$

$$\underline{\underline{C}}' = \underline{\underline{F}}'^T\underline{\underline{F}}' = (\underline{\underline{Q}}\,\underline{\underline{F}})^T(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underbrace{\underline{\underline{Q}}^T\underline{\underline{Q}}}_{I}\,\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow \underset{=}{C}$ is not affected by change in observer $\Rightarrow$ objective

material tensor C_{IJ} naturally objective

What about spatial tensors?

Axiom of frame indifference

Field ϕ, $\underline{w}$ and $\underset{=}{S}$ are called frame indifferent or objective if for all superposed rigid motions $\underline{x}' = \underset{=}{Q}\,\underline{x} + \underline{c}$ we have

$$\boxed{\begin{aligned} \phi'(\underline{x}', t) &= \phi(\underline{x}, t) \\ \underline{w}'(\underline{x}', t) &= \underset{=}{Q}\,\underline{w}(\underline{x}, t) \\ \underset{=}{S}'(\underline{x}', t) &= \underset{=}{Q}\,\underset{=}{S}(\underline{x}, t)\,\underset{=}{Q}^T \end{aligned}}$$

scalar field

vector field

tensor field

see lecture 6

Is spatial velocity gradient objective?

Lecture 16: $\underset{=}{\ell} = \nabla_x \underline{v} = \dot{\underset{=}{F}}\,\underset{=}{F}^{-1}$

$\underset{=}{\ell}' = \nabla_{x'} \underline{v}' = \dot{\underset{=}{F}}'\,\underset{=}{F}'^{-1}$

$$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}} \qquad\qquad \underline{\underline{\ell}}' \overset{?}{=} \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T$$

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\underbrace{\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}}_{\underline{\underline{\ell}}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underbrace{\underline{\underline{F}}\,\underline{\underline{F}}^{-1}}_{\underline{\underline{I}}}\,\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T \qquad \text{not objective}$$

$\Rightarrow$ that's why $\nabla_x \underline{v}$ is <u>not</u> used in constitutive laws

The non-objective term $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

it represents the rigid body angular velocity change between the observers.

HW9 $\rightarrow$ $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ is skew symmetric
$\rightarrow$ rotation

$\Rightarrow$ base our constitutive laws on symmetric part of $\underline{\underline{\ell}}$ $\qquad \underline{\underline{d}} = \mathrm{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$

$$\mathrm{sym}(\underline{\underline{\ell}}') = \underline{\underline{Q}}\,\mathrm{sym}(\underline{\underline{\ell}})\,\underline{\underline{Q}}^T + \cancel{\mathrm{sym}}(\underline{\underline{\Omega}})$$

$$\underline{\underline{d}}' = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws

Material frame indifferent functions

Field: $\phi(\underline{x}, t)$ scalar

$\underline{w}(\underline{x}, t)$

$\underline{\underline{S}}(\underline{x}, t)$

fields because they depend directly on $\underline{x}$

Constitutive functions are not fields

but they depend on fields as input.

internal energy: $u(\underline{x}, t) = \hat{u}(\rho(\underline{x}, t), \theta(\underline{x}, t)) = \hat{u}(\rho, \theta)$

$u \to$ field $\hat{u} \to$ is constitutive function

heat flow: $\underline{q}(\underline{x}, t) = \hat{\underline{q}}(\theta(\underline{x}, t)) = \hat{\underline{q}}(\theta)$

Cauchy stress: $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\rho(\underline{x},t), \theta(\underline{x},t), \underline{\underline{d}}(\underline{x},t))$

$$= \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}})$$

Constitutive functions: $\hat{u}(\rho,\theta)$, $\hat{\underline{q}}(\theta)$, $\hat{\underline{\underline{\sigma}}}(\rho,\theta,\underline{\underline{d}})$

$\hat{u} = \rho c_p \theta \qquad u(\underline{x},t) = u'(\underline{x}',t)$

As such constitutive functions are not directly dependent on frame but their input fields are

Consider two frames $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$

$$\underline{\underline{\sigma}}'(\underline{x}',t) = \hat{\underline{\underline{\sigma}}}(\underset{\rho'(\underline{x}')}{\underset{\uparrow}{\rho'}}, \theta', \underline{\underline{d}}') = \hat{\underline{\underline{\sigma}}}(\rho', \theta', \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T)$$

$$\sigma' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T$$

$$\boxed{\hat{\sigma}(\rho', \theta', \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T) = Q\,\hat{\underline{\underline{\sigma}}}(\rho,\theta,\underline{\underline{d}})\,\underline{\underline{Q}}^T}$$

condition for $\hat{\underline{\underline{\sigma}}}$ to be invariant/objective

Isotropic functions

Functions that are frame invariant are called isotropic.

functions:	$\hat{\phi}$	$\hat{\underline{w}}$	$\hat{\underline{\underline{\sigma}}}$
inputs:	θ	$\underline{v}$	$\underline{\underline{S}}$

Following requirements ~~of~~ for isotropic functions

$$\hat{\phi}(\theta) = \hat{\phi}(\theta) \qquad \hat{\phi}(\underline{\underline{Q}}\,\underline{v}) = \hat{\phi}(\underline{v}) \qquad \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\underline{\underline{S}})$$

$$\hat{\underline{w}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{w}}(\theta) \qquad \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{v}) \qquad \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{\underline{S}})$$

$$\hat{\underline{\underline{\sigma}}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\theta)\,\underline{\underline{Q}}^T \qquad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{v})\,\underline{\underline{Q}}^T \qquad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T$$

Examples:

1) $\hat{\phi}(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\hat{\phi}(\underbrace{\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T}_{\underline{\underline{S}}'}) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{S}})\det(\underline{\underline{Q}}^T)$$

$$= \det(S) = \hat{\phi}(\underline{\underline{S}})$$

$$\hat{\phi}(\underbrace{\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T}_{\underline{\underline{S}}'}) = \hat{\phi}(\underline{\underline{S}})$$

2) $\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\hat{\underline{u}}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = (\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T)(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underbrace{\cancel{\underline{\underline{Q}}^T\underline{\underline{Q}}}}_{\underline{\underline{I}}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{Q}}\,\hat{\underline{u}}(\underline{v}, \underline{\underline{A}})$$

Representation of isotropic tensorfunctions

An isotropic function $\underline{\underline{G}}(\underline{\underline{A}}): \mathcal{V}^2 \to \mathcal{V}^2$ that maps sym. tensors into sym. tensors must have the following form.

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(\mathcal{I}_A)\,\underline{\underline{I}} + \alpha_1(\mathcal{I}_A)\,\underline{\underline{A}} + \alpha_2(\mathcal{I}_A)\,\underline{\underline{A}}^2}$$

Rivlin-Ericksen Repres. Thm

where $\alpha_0, \alpha_1, \alpha_2$ are functions of the set of principal invariants of $\underline{\underline{A}}$

$$\mathcal{I}_A = \{ I_1(\underline{\underline{A}}), I_2(\underline{\underline{A}}), I_3(\underline{\underline{A}}) \}$$

- $\underline{\underline{G}}$ is sym. if $\underline{\underline{A}}$ is sym.
- To see $\underline{\underline{G}}$ is isotropic assume $\alpha_0, \alpha_1, \alpha_2 = \text{const}$

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0 \underset{\underset{\underline{\underline{Q}}\,\underline{\underline{Q}}^T}{\parallel}}{\underline{\underline{I}}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underbrace{\underline{\underline{Q}}^T\underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T$$

$$= \alpha_0\, \underline{\underline{Q}}\,\underline{\underline{I}}\,\underline{\underline{Q}}^T + \alpha_1\, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2\, \underline{\underline{Q}}\,\underline{\underline{A}}^2\,\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}} \left(\alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{A}} + \alpha_2 \underline{\underline{A}}^2 \right) \underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\, G(A) \underline{\underline{Q}}^T \quad \checkmark \qquad \text{isotrope/objective}$$

This is the most general form of a constitutive eqn.

Isotropic 4th order tensor

If $\underline{\underline{G}}(\underline{\underline{A}})$ is linear function

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\, \underline{\underline{A}}$$

where $\mathbb{C}$ is 4th order tensor

If we also require:

1) $\mathbb{C}\underline{\underline{A}}$ is sym. for every sym. $\underline{\underline{A}}$.

2) $\mathbb{C}\underline{\underline{W}} = \underline{\underline{0}}$ for every skew-sym. $\underline{\underline{W}}$

Then there are two scalars λ, μ such that

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\underline{\underline{A}} = \lambda \operatorname{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu \operatorname{sym}(\underline{\underline{A}})}$$

This follows from representation thm.

$$\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,\underline{\underline{A}} + \cancel{\alpha_2(I_A)\,\underline{\underline{A}}^2}$$

$\cdot$ where $I_A = \{\underbrace{\mathrm{tr}(A)}_{I_1}, \underbrace{\tfrac{1}{2}[\mathrm{tr}(\underline{\underline{A}})^2 - \mathrm{tr}(\underline{\underline{A}}^2)]}_{I_2}, \underbrace{\det \underline{\underline{A}}}_{I_3}\}$

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only $\mathrm{tr}(A)$ is a linear function

for $\underline{\underline{G}}(\underline{\underline{A}})$ to be linear in $\underline{\underline{A}}$:

$\alpha_2 = 0$ $\quad$ $\alpha_1 = \text{const.} = c_2$

$\alpha_0 = c_0\,\mathrm{tr}(\underline{\underline{A}}) + c_1$ $\quad$ $\alpha_1 = c_2$

where c_0, c_1, c_2 are const.

$\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \Rightarrow c_1 = 0$

set $c_0 = \lambda$ $\quad$ $c_2 = 2\mu$

$$\Rightarrow \underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu\,\underline{\underline{A}}$$

if $\underline{\underline{G}}(\underline{\underline{W}}) = 0$ $\quad$ $(\mathrm{tr}(\underline{\underline{W}}) = 0)$

$$\Rightarrow \boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu\,\mathrm{sym}(A)}$$

most general linear constitutive law!

Example linear elasticity : $\underline{\underline{A}} = \nabla \underline{u}$ $\quad \mathrm{tr}(\nabla u) = \nabla \cdot u$

$$\underline{\underline{\varepsilon}} = \lambda (\nabla \cdot \underline{u}) \underline{\underline{I}} + \mu (\nabla \underline{u} + \nabla \underline{u}^T) \quad \checkmark$$