Lecture 24: Newtonian Fluids

Logistics: - HW9 due Dec 1

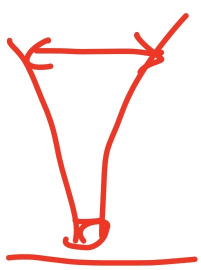
Last time: - Material constraints $\gamma(\underline{\underline{F}}(\underline{X},t)) = 0$

incompressibility: $\gamma(\underline{\underline{F}}) = \det(\underline{\underline{F}}) - 1$

$\dot{\gamma} = 0 \Rightarrow \nabla_x \cdot \underline{v} = 0$

- Stress field: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$

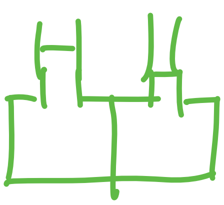
constraint　　　constitutive law



incompressibility: $\underline{\underline{\sigma}}^r = -p(\underline{x},t)\,\underline{\underline{I}}$

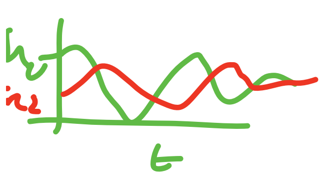
"pressure is a Lagrange multiplier that enforces the incompressibility"

Ideal fluids: $p = \text{const.}$　　$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}$



$$\Rightarrow \boxed{\begin{aligned} \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v} &= -\frac{1}{\rho_0}\nabla p + \underline{b} \\ \nabla_x \cdot \underline{v} &= 0 \end{aligned}}$$

Euler Eqns



$\Rightarrow \underline{\underline{\sigma}} : \underline{\underline{d}} = 0$　no stress power　no energy diss.

Bernoulli Thm & Irrotational motion

Today: Newtonian Fluids

Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Ref. mass is uniform $\rho_0(\underline{X}) = \rho_0$

2) Incompressible: $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress is Newtonian

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$$

$$= -p\,\underline{\underline{I}} + \mathbb{C}\,\nabla_x \underline{v}$$

because $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \Rightarrow$ left minor sym. $\mathbb{C}$

$$(\mathbb{C}\underline{\underline{A}})^T = \mathbb{C}\underline{\underline{A}}$$

trace condition $\operatorname{tr}(\mathbb{C}\underline{\underline{A}}) = 0$ if $\operatorname{tr}\underline{\underline{A}} = 0$

$$\Rightarrow \quad p = \frac{1}{3}\operatorname{tr}(\underline{\underline{\sigma}}) \qquad \operatorname{tr}(\underline{\underline{\sigma}}^a) = 0$$

$$\operatorname{tr}(\nabla_x \underline{v}) = \nabla \cdot \underline{v} = 0$$

General linear isotropic function

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda \operatorname{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu \operatorname{sym}(\underline{\underline{A}})$$

$$\underline{\underline{A}} = \nabla_x \underline{v}$$

$$\underline{\underline{\sigma}}^a = \lambda \operatorname{tr}(\nabla_x \underline{v})\,\underline{\underline{I}} + 2\mu \operatorname{sym}(\nabla_x \underline{v})$$

shear viscosity

$$= \lambda \cancel{\nabla_x \cdot \underline{v}}\, \underline{\underline{I}} + \mu \left(\nabla_x \underline{v} + \nabla_x \underline{v}^T \right)$$ σ

$$\underline{\underline{\sigma}}^a = \mu \left(\nabla \underline{v} + \nabla \underline{v}^T \right) = 2\mu \, \mathrm{sym}(\nabla v)$$

Put together:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a = -p \, \underline{\underline{I}} + \mu \left(\nabla \underline{v} + \nabla \underline{v}^T \right)$$

Last time: showed that $\underline{\underline{\sigma}}^r$ is objective

Is the $\underline{\underline{\sigma}}^a$ objective?

$$\underline{\underline{\sigma}}^a = 2\mu \, \mathrm{sym}(\nabla_x v) = 2\mu \, \underline{\underline{d}} \qquad \underline{\underline{d}} = \underline{\underline{d}}^T$$

Check objectivity: $\underline{x}' = Q \underline{x} + c \qquad \underline{\underline{d}}^* = \underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^T$

$$\underline{\underline{\sigma}}^{a\prime} = 2\mu \, \underline{\underline{d}}' = 2\mu \, \underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^T = \underline{\underline{Q}} \underbrace{(2\mu \underline{\underline{d}})}_{\underline{\underline{\sigma}}^a} \underline{\underline{Q}}^T = \underline{\underline{Q}}\, \underline{\underline{\sigma}}^a \underline{\underline{Q}}^T$$

Check left minor sym:

$$(\mathbb{C} \, \nabla_x \underline{v})^T = (2\mu \underline{\underline{d}})^T = 2\mu \underline{\underline{d}}^T = 2\mu \underline{\underline{d}} = \mathbb{C} \, \nabla v$$

Check trace condition

$$\mathrm{tr}(\mathbb{C} \, \nabla \underline{v}) = \mathrm{tr}(2\mu \underline{\underline{d}}) = 2\mu \, \mathrm{tr}(\underline{\underline{d}}) = 0 \quad \text{if} \quad \mathrm{tr}(\underline{\underline{d}}) = \underline{0}$$

Navier-Stokes equs

Set $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu(\nabla\underline{v} + \nabla\underline{v}^T)$

subst into lin. mom. balance.

$$\rho_0 \dot{\underline{v}} = \nabla\cdot\underline{\underline{\sigma}} + \rho_0 \underline{b}$$

$$= \nabla\cdot[-p\underline{\underline{I}} + \mu(\nabla\underline{v} + \nabla\underline{v}^T)] + \rho_0\underline{b}$$

if $\mu =$ const. we can simplify

$$\nabla\cdot\underline{\underline{\sigma}} = -\nabla p + \mu\,\nabla\cdot(\nabla v) + \mu\,\cancel{\nabla\cdot(\nabla v^T)}^{\,0} + \rho_0\underline{b}$$

$$\nabla\cdot\nabla v = \nabla^2\underline{v} = v_{i,jj}\,\underline{e}_i$$

$$\nabla\cdot(\nabla v)^T = v_{j,ij}\,\underline{e}_i = \underbrace{v_{j,ji}}_{\nabla\cdot v = v_{j,j}}\,\underline{e}_i = \nabla(\cancel{\nabla\cdot\underline{v}})^{0} = 0$$

$$\nabla\cdot\sigma = -\nabla p + \mu\nabla^2\underline{v}$$

so that

$$\boxed{\begin{aligned} \rho_0\left[\frac{\partial\underline{v}}{\partial t} + (\nabla\underline{v})\underline{v}\right] &= \mu\nabla^2\underline{v} - \nabla p + \rho_0\underline{b} \\ \nabla\cdot\underline{v} &= 0 \end{aligned}}$$

Navier-Stokes equ

$$\frac{\partial\underline{v}}{\partial t} + \underbrace{(\nabla v)\underline{v}}_{(\underline{v}\cdot\nabla)\underline{v}} = \underbrace{\frac{\mu}{\rho_0}}_{\nu}\nabla^2\underline{v} - \frac{\nabla p}{\rho_0} + \underline{b}$$

kinematic vel. ✓

Stress power of Newtonian fluid

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-p\underline{\underline{I}} + 2\mu\, \underline{\underline{d}}) : \underline{\underline{d}} = -p\, \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla\cdot \underline{v} = 0} + 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

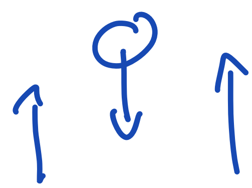
as last time

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = 2\mu\, \underline{\underline{d}} : \underline{\underline{d}} > 0 \quad \text{if} \quad \mu > 0$$

$\Rightarrow$ only if $\mu > 0$ energy is dissipated

Kinetic Energy of Fluid Motion

Dissipation of Kinetic Energy in Ideal and Newtonian fluids



First some useful results:

1) Integration by parts in fixed domain Ω with "no slip" boundaries $\underline{v} = \underline{0}$ on $\partial\Omega$

$$\boxed{\int_\Omega (\nabla^2 \underline{v}) \cdot \underline{v}\, dV = -\int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v})\, dV}$$

$\underline{\underline{A}}\,\underline{x} = A_{ij} x_j$

$\underline{\underline{A}}^T \underline{x} = A_{ji} x_j$

To see this:

$$(v_{i,j} v_i)_{,j} = \underline{v_{i,jj}}\, v_i + v_{i,j} v_{i,j}$$

$$v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j} \qquad A_{ij} B_{ij} = A : B$$

$$(\nabla^2 \underline{v})\cdot \underline{v} = \underline{\nabla\cdot((\nabla \underline{v})^T \underline{v})} - \nabla \underline{v} : \nabla \underline{v}$$

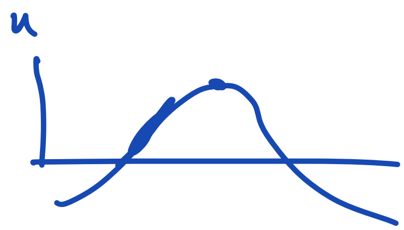
vec substitute and use Div Thm

$$\int_\Omega (\nabla^2 \underline{v})\cdot \underline{v}\, dV = \int_{\partial\Omega} \left((\nabla v)^T \overset{0}{\cancel{\underline{v}}}\right)\cdot \underline{n}\, dA - \int_\Omega \nabla \underline{v} : \nabla v\, dV$$

2) Poincaré Inequality

$$\boxed{\| \underline{u} \|_\Omega \le \lambda \| \nabla \underline{u} \|_\Omega}$$

u ∇u



Notice λ has units of L^2 and scales with the area of Ω

Consider a domain Ω with $\underline{v} = 0$ on $\partial\Omega$ and a conservative body force: $\underline{b} = -\nabla\psi$.

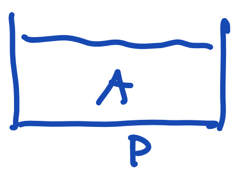
Kinetic Energy: $K(t) = \int_\Omega \frac{1}{2} \rho_0 |\underline{v}|^2\, dV \qquad K(0) = K_0$

I) Newtonian fluid

$$\boxed{K(t) \le K_0 \, e^{-2\mu t/\lambda \rho_0}}$$

$\nu = \frac{\mu}{\rho_0}$ ~~dynamic~~ kinematic viscosity

kinetic energy of a newtonian fluid dissipates to zero exponentially fast



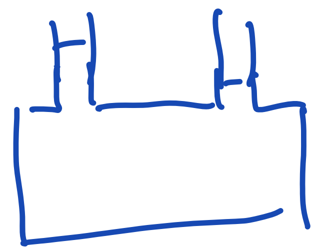
channel flow

$$\bar{v} \sim \frac{A}{P}$$

II ideal fluid

$\mu = 0$

$$K(t) = K_0$$



By definition

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho_0 \frac{d}{dt}|\underline{v}|^2 \, dV = \int_\Omega \rho_0 \, \dot{\underline{v}} \cdot \underline{v} \, dV_x$$

from N-S: $\rho_0 \dot{\underline{v}} = \mu \nabla^2 \underline{v} - \nabla \varphi$

substitute:

$$\frac{d}{dt} K(t) = \int_\Omega \left(\mu \nabla^2 \underline{v} - \nabla\psi\right) \cdot \underline{v} \, dV_x$$

show $\int_\Omega \nabla\psi \cdot v \, dV = 0$

$$\nabla \cdot (\psi \underline{v}) = \nabla\psi \cdot \underline{v} + \psi \cancel{\nabla \cdot \underline{v}}^{\,0}$$

$$\int_\Omega \nabla \cdot (\psi \underline{v}) \, dV = \int_{\partial\Omega} \psi \cancel{\underline{v}}^{\,0} \cdot \underline{n} \, dA = 0$$

$$\frac{d}{dt} K(t) = \int_\Omega \mu \nabla^2 \underline{v} \cdot \underline{v} \, dV$$

$$= -\mu \int_\Omega \nabla v : \nabla v \, dV$$

$$\|\nabla v\|$$

for ideal fluid $\mu = 0 \Rightarrow K(t) = \text{const} = K_0$

for Newtonian fluid appy Poincare inequality

$$\frac{1}{\lambda} \int_\Omega |\underline{v}|^2 dV \leq \lambda \int_\Omega \nabla \underline{v} : \nabla \underline{v} \, dV$$

substitute

$$\frac{d}{dt} K(t) \leq -\frac{\mu}{\lambda} \int_\Omega |\underline{v}|^2 dV = -\frac{2\mu}{\lambda \rho_0} K(t)$$

$$K = \frac{1}{2} \int_\Omega \rho_0 |v|^2 dV$$

$\Rightarrow$ ODE

$$\boxed{\frac{dK}{dt} \le -\frac{2\mu}{\rho_0 \lambda} K}$$

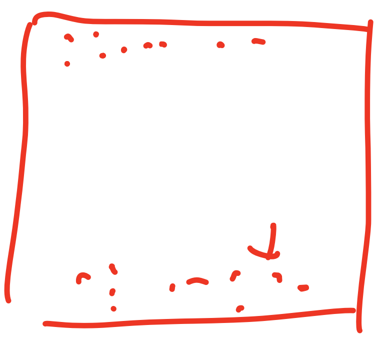
integrate by sep. of parts

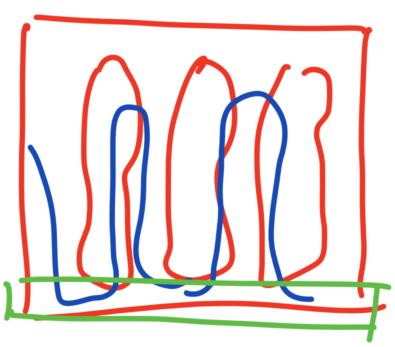
$$\Rightarrow \quad K(t) \le K_0 \, e^{-\frac{2\mu}{\lambda \rho_0} t}$$

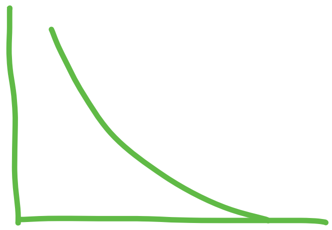
in absence of fluid motion on boundary

velocity decays exponentially.

rate of decay $\boxed{\nu = \frac{\mu}{\rho_0}}$ kinematic viscosity







$x \sim \exp(t)$

Stevenson