Lecture 25: Stokes flow

Logistics: - HW9 due Th.

Projects due Friday 9th Dec

Last times: - Newtonian fluids

- stress: $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mu(\nabla \underline{v} + \nabla \underline{v}^T)$ linear

- if $\mu = \text{const}$: $\Rightarrow$ Navier-Stokes Equs

$$\rho_0\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right] = \mu \nabla^2 \underline{v} - \nabla p + \rho b$$

$$\nabla \cdot \underline{v} = 0$$

- Stress power in a Newtonian fluid

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = 2\mu\, \underline{\underline{d}} : \underline{\underline{d}} \qquad \underline{\underline{d}} = \tfrac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$$

$$\Rightarrow \mu > 0$$

- Kinetic Energy in flow

$$K(t) \leq K_0\, e^{-2\mu t/\lambda \rho_0}$$

Today: - Scaling N-S equ

- Rayleigh's problem

- Stokes equ.

Scaling the N-S equations

lin. mom: $\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v} = \mu \nabla^2 \underline{v} - \nabla p + \rho \underline{g}$

continuity: $\nabla \cdot \underline{v} = 0$

Reduced pressure: $\underline{g} = -g\hat{z}$ $\quad g = |\underline{g}|$ $\quad \hat{z}$ = unit upward

$$-\nabla p + \rho \underline{g} = -\nabla p - \rho g \hat{z} = -\nabla(\underbrace{p + \rho g z}_{\pi}) = -\nabla \pi$$

$\nabla z = \hat{z}$

so that we have:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right) - \mu \nabla^2 \underline{v} = -\nabla \pi$$

Equation with 4 different terms.

Q: How important is each term?

Non-dimensionalize with generic quantities to define standard dim. less parameters:

- Dependent variables: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parameters: $\rho \left[\frac{M}{L^3}\right]$ $\quad \mu \left[\frac{M}{LT}\right]$ $\rightarrow$ $\nu = \frac{\mu}{\rho} \left[\frac{L^2}{T}\right]$

+ Geometry, BC, IC

Use parameters to scale the variables

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

$\underline{v} = v_c \underline{v}'$

here v_c, π_c, x_c & t_c are char. scales that are problem dependent and formed by parameters.
and constant

Substitute definitions into PDE

$$\frac{\partial \underline{v}}{\partial t} = \frac{\partial (v_c \underline{v}')}{\partial (t_c t')} = \frac{v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} \qquad \nabla \approx \frac{\partial}{\partial x}$$

$$\frac{\rho v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c}(\nabla' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla' \pi'$$

Option 1: Scale to accumulation term

$$\underbrace{\frac{\partial \underline{v}'}{\partial t'}}_{\text{dimless}} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1}(\nabla' \underline{v}')\underline{v}' - \underbrace{\frac{\mu t_c}{\rho x_c^2}}_{\Pi_2}\nabla'^2 \underline{v}' = -\underbrace{\frac{\pi_c t_c}{x_c \rho v_c}}_{\Pi_3}\nabla' \pi'$$

$$Pa = \frac{F}{A} = \frac{ML}{L^2T^2} = \frac{M}{LT^2}$$

Three dimensionless groups:

$$\Pi_1 = \frac{v_c t_c}{x_c} \qquad \Pi_2 = \frac{\nu t_c}{x_c^2} \qquad \Pi_3 = \frac{\pi_c t_c}{x_c \rho v_c}$$

$$\frac{L}{T}\frac{T}{L} = 1 \qquad \frac{L^2}{T}\frac{T}{L^2} = 1 \qquad \frac{\cancel{M}}{L\cancel{T^2}}\frac{\cancel{T}}{\cancel{L}}\frac{L^3}{\cancel{M}}\frac{\cancel{T}}{L} = 1$$

Use Π's to define time scales:

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective time scale $\quad t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1 \Rightarrow$ diffusive time scale $\quad t_c = t_D = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale:

$$\Pi_3 = \frac{\pi_c t_c}{x_c \rho v_c} = 1 \Rightarrow \quad \pi_c = \frac{x_c \rho v_c}{t_c}$$

Choose a diffusive time scale: $\quad \Pi_2 = 1 \quad \Pi_3 = 1$

$$\Pi_1 = \frac{v_c t_c}{x_c} \overset{t_c = t_D}{=} \frac{v_c x_c^2}{x_c \nu} = \frac{v_c x_c}{\nu} = Pe_m = Re$$

Peclet number Pe: $\frac{\text{adv}}{\text{diff}}$ transport

Reynolds number: $Re = \frac{v_c x_c}{\nu}$

Substituting

$$\frac{\partial \underline{v}'}{\partial t'} + \underset{\substack{\uparrow \\ \text{one parameter}}}{Re}\, \cancelto{0}{(\nabla' \cdot \underline{v}')\, \underline{v}'} - \nabla'^2 \underline{v}' = -\nabla' \pi'$$

Adv. mom. transport vanishes as $Re \to 0$

For viscous flow of a glacier:

$\rho = 10^3 \frac{kg}{m^3} \qquad v_c = 100 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\mu = 10^4$ Pa s $\qquad x_c = 10^2$ m (thickness)

$$Re = \frac{v_c x_c \rho}{\mu} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

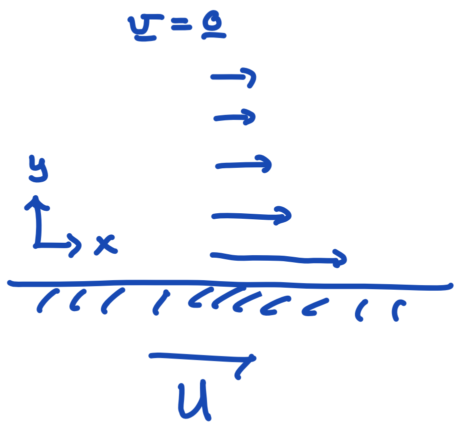
$\Rightarrow$ adv. mom. transport is negligible

Mom. balance simplifies

$$\boxed{\frac{\partial \underline{v}'}{\partial t'} - \nabla'^2 \underline{v}' = -\nabla' \pi' \quad \& \quad \nabla' \cdot \underline{v}' = 0} \qquad \text{linear}$$

Rayleigh's problem

- semi-infinite half-space
- stationary fluid
- Impulsively started plate with velocity U



$v_c = U \qquad Re = \frac{U x_c \rho}{\mu} \ll 1$

What is x_c? not obvious

assume $\quad x_c \ll \frac{\mu}{U\rho} \quad \rightarrow \quad Re \ll 1$

Redimensionalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \binom{u}{w}$$

Simplify equations further:

1) Flow is horizontal: $w = 0 \qquad \underline{v} = \binom{u}{0}$

2) From continuity: $\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial w}{\partial y}} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$

$\Rightarrow u = u(y)$

3) Domain is infinite in x-dir; but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

$$\nabla^2 \underline{v} = v_{i,jj}\, \underline{e}_i \qquad i,j \in \{1,2\}$$

$$= \begin{pmatrix} v_{1,11} + v_{1,22} \\ v_{2,11} + v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancelto{0}{u_{xx}} + u_{yy} \\ \cancelto{0}{w_{xx}} + \cancelto{0}{w_{yy}} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting:

x-mom.: $\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0$

y-mom.: $0 - 0 = -\frac{\partial \pi}{\partial y}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

IC: $u(t=0, y) = 0$

BC: $u(t, y=0) = U$

$\Rightarrow$ diffusion equation

similar to heating the end of a semi-inf. rod.

$\nu \quad \left[\frac{L^2}{T}\right] \qquad \nu t \quad [L^2] \qquad \sqrt{\nu t} \quad [L]$

why not choose $x_c = \sqrt{\nu t}$

Introducing a self-similar variable

$\eta = \frac{y}{\sqrt{4\nu t}}$ $\eta = \eta(x,t)$ dep. variable

$f(\eta) = \frac{u}{U}$ dim. velocity

Reduce our PDE to an ODE

$$u(x,t) \longrightarrow f(\eta)$$

$\Rightarrow$ similarity solution:

derivatives: $\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$ $\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$

The derivatives of u transform as:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t}\frac{df}{d\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$$

substituting into PDE:

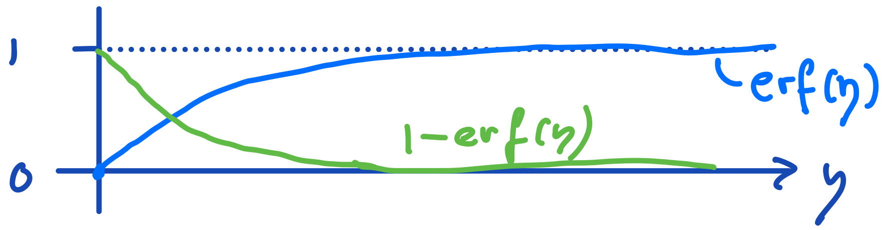
$$\boxed{\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0}$$

BC: $f(\eta = 0) = 1$

$f(\eta \to \infty) = 0$

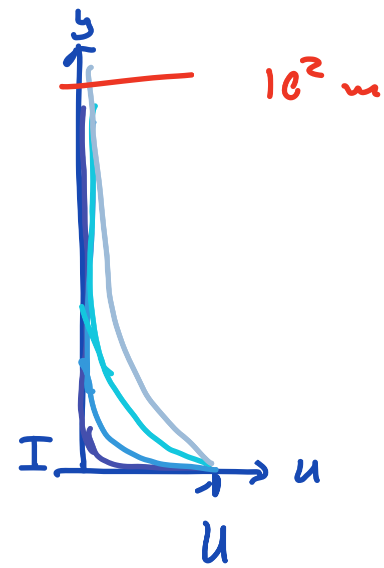
Self-similar ODE

Solution: $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)



Resubstitute self-similar variables:

$$\boxed{u(y,t) = U\left[1-\mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]}$$



Diffusive boundary layer where mom. added by the boundary cond. has penetrated $\sqrt{4\nu t}$ into the fluid. $\delta(t) \sim t^{1/2}$

Is it worth resolving this transient?

$$t_D = \frac{x_c^2}{\nu} = \frac{x_c^2 \rho}{\mu} = 10^{4+3-14}\,s = 10^{-7}\,s$$

negligible to any time scales of interest re glacier

Stokes Equation

Scale to viscous term

$$\frac{\rho v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c}(\nabla' \underline{v}')\underline{v}' - \underline{\underline{\frac{\mu v_c}{x_c^2}\nabla'^2 \underline{v}'}} = -\frac{\pi_c}{x_c}\nabla' \pi'$$

divide through by $\frac{\mu v_c}{x_c^2}$

$$\frac{x_c^2}{\nu t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{v_c x_c}{\nu}(\nabla' \underline{v}')\underline{v}' - \nabla'^2 \underline{v}' = -\underbrace{\frac{\pi_c x_c}{\mu v_c}}_{1}\nabla' \pi'$$

$$1 \Rightarrow \pi_c = \frac{\mu v_c}{x_c}$$

choose an advective time scale

$$t_c = t_A = \frac{x_c}{v_c}$$

$$\underline{\underline{Re}}\left(\frac{\partial \underline{v}'}{\partial t'} + (\nabla' \underline{v}')\underline{v}'\right) - \nabla'^2 \underline{v}' = -\nabla' \pi'$$

limit $Re \to 0$ we obtain

$$\nabla'^2 \underline{v}' = \nabla' \pi'$$

$$\nabla' \cdot \underline{v}' = 0$$

Redimensionalize

$$\boxed{\begin{aligned} \mu \nabla^2 \underline{v} &= \nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Stokes Equ

linear