Lecture 26: Power-law creep

Logistics: - HW 9 due
- Last class
- Projects due Fri Dec 9th

Office hrs schedule stays for next week

Last time: - Stokes flow

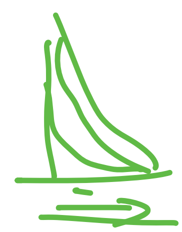
- Scaling of NS eqn.

⇒ Reynolds number: $\boxed{Re = \frac{vL}{\nu}}$

- Option 1: Transient linear equ

$$\frac{\partial v}{\partial t} - \nu \nabla^2 v = \cancel{-\nabla \pi}$$

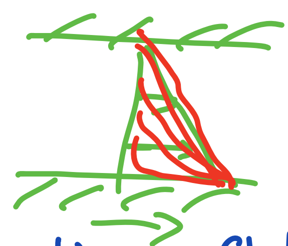
⇒ Reynolds problem



- Option 2: Steady → Stokes Equ (linear

$$\nu \nabla^2 \underline{v} = \nabla \pi$$

$$\nabla \cdot \underline{v} = 0$$



Today: Power-law creep → non-linear Stokes

Power law creep and non-Newtonian viscosity

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + C\,\nabla\underline{v}$ linear

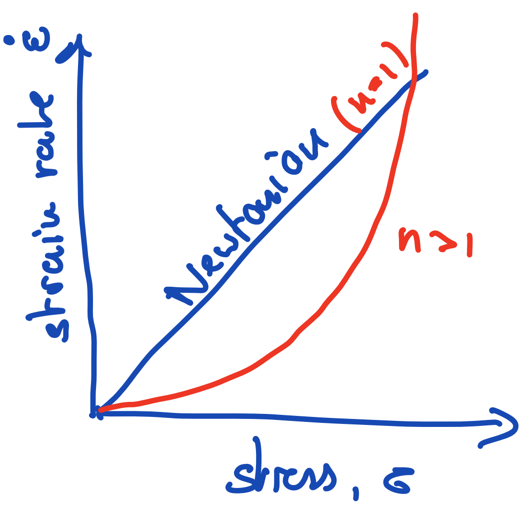
Note: Constitutive law is linear
but the mom. bal. is non-linear
due to advective term $(\nabla\underline{v})\underline{v}$

In Earth science most important non-New. rheology is power-law creep:

$\dot{\varepsilon} = \frac{1}{2}(\nabla v + \nabla^T v)$

Newt: $\underline{\underline{\sigma}} = -pI + 2\mu\,\underline{\underline{\dot{\varepsilon}}}$

$\sigma \sim \dot{\varepsilon}$



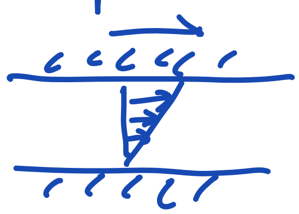
Experimentally we often see:

$$\boxed{\dot{\varepsilon} = A\,\sigma^n}$$

n = stress exponent $n \geq 1$

Note: This is scalar relationship!
How do you extend this to tensor form?

Simple shear



$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} = \dot{\underline{\underline{\varepsilon}}} = \begin{bmatrix} 0 & \dot\varepsilon_s & 0 \\ \dot\varepsilon_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

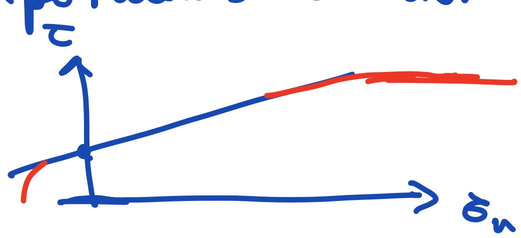
$$\dot\varepsilon_s = A\,\sigma_s^n$$

A is function of T ...

n = stress exponent is constant
(at least as long as def. mech. is same)

What is general tensor form of power-law creep?

1) Experiments are not affected by pressure



$$\tau = c_0 + \mu\,\sigma_n$$

$\Rightarrow$ use deviatoric stress & strain rate

$$\underline{\underline{\sigma}} = \mathrm{sph}(\underline{\underline{\sigma}}) + \underbrace{\mathrm{dev}(\underline{\underline{\sigma}})}$$

$$= \mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}} + \underline{\underline{\sigma}}'$$

$$\underline{\underline{\sigma}}' = \underline{\underline{\sigma}} - \mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$$

$$\mathrm{tr}(\underline{\underline{\sigma}}') = 0$$

2) Objective $\Rightarrow$ invariants of $\dot{\underline{\underline{\varepsilon}}}$

$$I_1(\underline{\underline{S}}) = tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 = S_{11} + S_{22} + S_{33}$$
$$I_2(\underline{\underline{S}}) = \frac{1}{2}\left(tr(\underline{\underline{S}})^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$
$$I_2(\underline{\underline{S}}) = -(S_{11}S_{22} + S_{11}S_{33} + S_{22}S_{33}) + S_{12}^2 + S_{13}^2 + S_{23}^2$$
$$I_3(\underline{\underline{S}}) = det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3$$

We can only use $I_2(S)$ because $I_1(S) = I_3(S) = 0$

Invariants of deviatoric tensors $\underline{\underline{\sigma}}'$ $\dot{\underline{\underline{\varepsilon}}}'$

$$J_1(\underline{\underline{\sigma}}) = J_1(\dot{\underline{\underline{\varepsilon}}}) = 0$$

$$\boxed{J_2(\underline{\underline{\sigma}}) = \frac{1}{2}\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} \qquad J_2(\dot{\underline{\underline{\varepsilon}}}) = \frac{1}{2}\dot{\underline{\underline{\varepsilon}}}' : \dot{\underline{\underline{\varepsilon}}}'$$

$$J_2(\underline{\underline{\sigma}}) = \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{13}^2$$
$$\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}' = \sigma_{12}'^2 + \sigma_{21}'^2 + \sigma_{13}'^2 + \sigma_{31}'^2 + \sigma_{23}'^2 + \sigma_{32}'^2 = 2J_2(\underline{\underline{\sigma}})$$
$$\underline{\underline{\sigma}}' = \underline{\underline{\sigma}}'^{\,t}$$

For $\underline{\text{incomp.}}$ material: $J_2(\dot{\underline{\underline{\varepsilon}}}) = I_2(\dot{\underline{\underline{\varepsilon}}}) = \frac{1}{2}\dot{\underline{\underline{\varepsilon}}} : \dot{\underline{\underline{\varepsilon}}}$

$$\nabla \cdot v = tr(\nabla \underline{v}) = 0$$

$$\dot{\underline{\underline{\varepsilon}}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^T\right)$$

We can define effective stress & strain rate

$$\sigma'_E = \sqrt{\frac{1}{2}\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} \qquad \dot{\varepsilon}_E = \sqrt{\frac{1}{2}\,\dot{\underline{\underline{\varepsilon}}} : \dot{\underline{\underline{\varepsilon}}}}$$

rewrite the scalar power law as:

— scalar
— tensor

$$\dot{\varepsilon}_E = A\,{\sigma'_E}^{\,n}$$

because $\dot{\varepsilon}_E$ & σ'_E are invariants this relation is automatically objective

To extend this to tensorial form we assume: $\dot{\underline{\underline{\varepsilon}}} = \lambda(\sigma_E)\,\underline{\underline{\sigma}}'$

$$\dot{\varepsilon}_E = \sqrt{\frac{1}{2}\,\dot{\underline{\underline{\varepsilon}}} : \dot{\underline{\underline{\varepsilon}}}} = \sqrt{\frac{1}{2}\lambda^2\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} = \lambda\sqrt{\frac{1}{2}\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'}$$

$$\dot{\varepsilon}_E = \lambda\,\sigma'_E$$

Now we have two relations

$$\dot{\varepsilon}_E = A\,{\sigma'_E}^{\,n} \quad \text{and} \quad \dot{\varepsilon}_E = \lambda\,\sigma'_E$$

equate: $\lambda\sigma'_E = A\,{\sigma'_E}^{\,n} \Rightarrow \lambda = A\,{\sigma'_E}^{\,n-1}$

Substitute into tensorial form

$$\boxed{\underline{\underline{\dot{\varepsilon}}} = A\,\sigma_E'^{(n-1)}\,\underline{\underline{\sigma}}'}$$

deviatoric stress term

$$\sigma_E = \sqrt{J_2(\underline{\underline{\sigma}})}$$

Compare to Representation theorem

$$\underline{\underline{\dot{\varepsilon}}}(\underline{\underline{\sigma}}') = \alpha_0(I_{\sigma'})\,\underline{\underline{I}} + \alpha_1(I_{\sigma'})\,\underline{\underline{\sigma}}' + \alpha_2(I_{\sigma'})\,\underline{\underline{\sigma}}'^2$$

$$\alpha_0 = \alpha_2 = 0 \qquad \alpha_1 = A\,\sigma_E'^{(n-1)} = \alpha_1\big(J_2(\underline{\underline{\sigma}})\big) \checkmark$$

$\Rightarrow$ frame invariant.

$$\alpha_1 = A\sqrt{J_2(\underline{\underline{\sigma}})}^{\,n-1} = A\,J_2(\underline{\underline{\sigma}})^{\frac{n-1}{2}}$$

Example: simple shear

$$\underline{\underline{\sigma}} = \begin{pmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\dot{\varepsilon}}} = \begin{pmatrix} 0 & \dot{\varepsilon}_s & 0 \\ \dot{\varepsilon}_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\underline{\sigma}} = \underline{\underline{\sigma}}' \qquad \underline{\underline{\dot{\varepsilon}}} = \underline{\underline{\dot{\varepsilon}}}'$$

$$\sigma_E' = \sqrt{\tfrac{1}{2}\,\underline{\underline{\sigma}} : \underline{\underline{\sigma}}} = \sqrt{\tfrac{1}{2}\left(\sigma_s^2 + \sigma_s^2\right)} = \sigma_s$$

$$\dot{\varepsilon}_E' = \dot{\varepsilon}_s$$

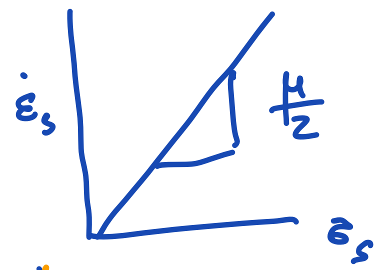
full tensorial: $\underline{\underline{\dot{\varepsilon}}} = A\,\sigma_E'^{\,n-1}\,\underline{\underline{\sigma}} \xrightarrow{\sigma_s\ \dot{\varepsilon}_s} \Rightarrow \dot{\varepsilon}_s = A\,\sigma_s^{\,n-1}\,\sigma_s = \underline{\underline{A\,\sigma_s^{\,n}}}$

Effective viscosity of power-law creep

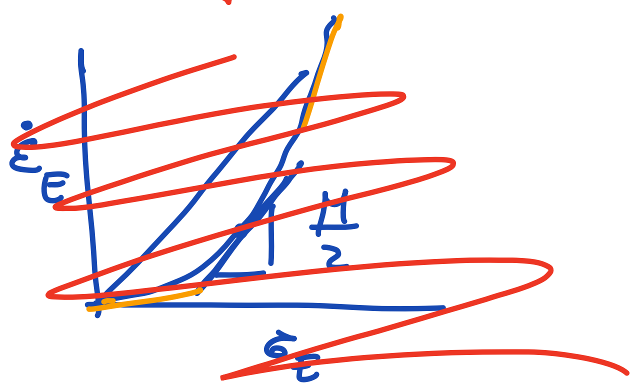
Standard Newtonian fluid

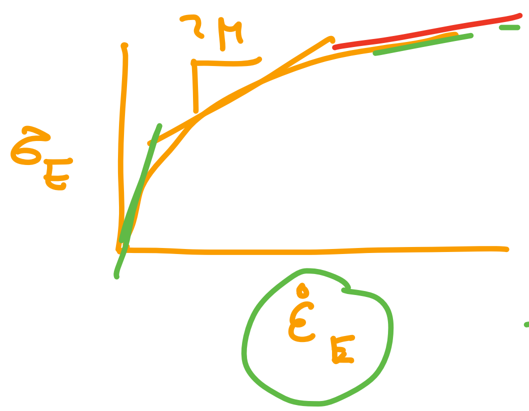
$$\underline{\underline{\sigma}}' = 2\mu\, \underline{\underline{\dot{\varepsilon}}}'$$

$$\sigma_s = 2\mu\, \dot{\varepsilon}_s \quad \Rightarrow \quad \boxed{\mu = \frac{\sigma_s}{2\dot{\varepsilon}_s}}$$



$$2\mu_E = \frac{\sigma_E}{\cancel{2}\,\dot{\varepsilon}_E}$$

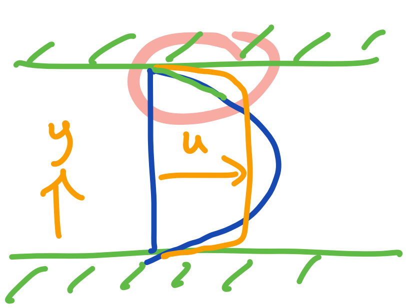




$\Rightarrow$ strain rate weakening

$\Rightarrow$ localization

Pipe flow



$$\dot{\varepsilon}_E = \dot{\varepsilon}_s = \frac{du}{dy}$$