Lecture 3: Tensor Algebra & Properties

Logistics: - HW 1 is due Th 2pm on Canvas

Last time: • Frame identities

$$\underline{e}_j = \delta_{ij}\,\underline{e}_i \qquad \underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\,\underline{e}_k$$

• $\epsilon\delta$ - identities

$$\epsilon_{pqs}\,\epsilon_{nrs} = \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn}$$

$$\epsilon_{pqs}\,\epsilon_{rqs} = 2\,\delta_{pr}$$

→ useful in deriving identities (HW)

• 2^{nd}-order tensors

Linear operators: $\boxed{\underline{v} = \underline{\underline{A}}\,\underline{u}}$

Dyadic product: $\boxed{(\underline{a} \otimes \underline{b})\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}}$

$$\underline{a} \otimes \underline{b} = \begin{bmatrix} a_1 b_1 & a_2 b_2 & a_3 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_3 & a_3 b_3 \end{bmatrix}$$

Basis for $\mathcal{V}^2$: $\underline{\underline{S}} = S_{ij}\,\underline{e}_i \otimes \underline{e}_j$

$$S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j$$

Today: • Tensor algebra in index notation
• Transpose + Sym.-Skew decomp.
• Trace + Spherical-Deviatoric decomp.
• Tensor scalar product
• Determinant & inverse
• Orthogonal tensors & change in basis
• Projection & Reflection tensors

Tensor algebra in index notation

$$\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$$

$$H_{ij}\, \underline{e}_i \otimes \underline{e}_j = S_{ij}\, \underline{e}_i \otimes \underline{e}_j + T_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

$$= (S_{ij} + T_{ij})\, \underline{e}_i \otimes \underline{e}_j$$

$$H_{ij} = S_{ij} + T_{ij}$$

$$\underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{T}}$$

$$= S_{ij}\, \underline{e}_i \otimes \underline{e}_j\, T_{kl}\, \underline{e}_k \otimes \underline{e}_l$$

$$= S_{ij}\, T_{kl}\, \boxed{(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)}$$

$$S_{ij}T_{kl}\ \boxed{\underbrace{(\underline{e}_i\otimes\underline{e}_j)(\underline{e}_k\otimes\underline{e}_l)}_{(\underline{e}_j\cdot\underline{e}_k)\,\underline{e}_i\otimes\underline{e}_l}} \Rightarrow \text{HW2}$$

$$= S_{ij}T_{kl}\ \delta_{jk}\,(\underline{e}_i\otimes\underline{e}_l)$$

$$= S_{ij}T_{jl}\,(\underline{e}_i\otimes\underline{e}_l)$$

$$H_{il}(\underline{e}_i\otimes\underline{e}_l) = S_{ij}T_{jl}\,(e_i\otimes\underline{e}_l) \leftarrow$$

$$\Rightarrow \boxed{H_{il} = S_{ij}T_{jl}}$$

$$\underline{\underline{A}}^T\underline{\underline{B}} = A_{ji}B_{jl}\,(\underline{e}_i\otimes\underline{e}_l)$$

$$\underline{\underline{A}}\,\underline{\underline{B}}^T = A_{ij}B_{lj}\,(\underline{e}_i\otimes\underline{e}_l)$$

Transpose of tensor

Definition

$$\boxed{\underline{\underline{S}}\,\underline{u}\cdot\underline{v} = \underline{u}\cdot\underline{\underline{S}}^T\underline{v}} \quad \text{for all } \underline{u},\underline{v}\in V^e$$

Transpose $\quad S^T_{ij} = S_{ji} \quad [\times]$

last time: $\underline{\underline{A}}\,\underline{v} = A_{ij} v_j \underline{e}_i$

$$(S_{ij} u_j \underline{e}_i)\cdot(v_\ell \underline{e}_\ell) = (u_k \underline{e}_k)\cdot(S^T_{ij} v_j \underline{e}_i)$$

$$S_{ij} u_j v_\ell \underbrace{(\underline{e}_i \cdot \underline{e}_\ell)} = S^T_{ij} u_k v_j (\underline{e}_k \cdot \underline{e}_i)$$

$$S_{ij} u_j v_\ell \quad \delta_{i\ell} = S^T_{ij} u_k v_j \quad \delta_{ki}$$

$$S_{ij} u_j v_i = S^T_{ij} u_i v_j$$

~~rename~~ flip i & j on rhs

$$S_{ij} \underline{u_j v_i} = S^T_{ji} \underline{u_j v_i}$$

$$\Rightarrow \quad S^T_{ji} = S_{ij} \qquad \boxed{S^T_{ij} = S_{ji}}$$

Properties:

1) $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$
2) $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$
3) $(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$

$\underline{\underline{S}}$ is symmetric if $\quad \underline{\underline{S}} = \underline{\underline{S}}^T \quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew-sym. if $\quad \underline{\underline{S}} = -\underline{\underline{S}}^T \quad S_{ij} = -S_{ji}$

Symmetric–Skew decomposition

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}} \qquad \begin{aligned} &\rightarrow \underline{\underline{E}} = \underline{\underline{E}}^T \\ &\rightarrow \underline{\underline{W}} = -\underline{\underline{W}}^T \end{aligned}$$

Note: $\quad \underline{\underline{W}} = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$ only 3 indep. comp.

any skew. sym. tensor has an axial vector

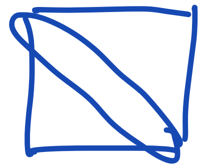
$\boxed{\underline{\underline{W}}\,\underline{v} = \underline{w} \times \underline{v}}$ for any $\underline{v} \in \mathcal{V}$

Relation: $\quad W_{ij} = -\epsilon_{ijk} w_k$

$$w_k = -\tfrac{1}{2}\epsilon_{ijk} W_{ij} \qquad \underline{\underline{W}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Trace of a tensor



Trace of a dyadic product

$$\boxed{\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b}} = a_i b_i$$

this implies: $\boxed{\mathrm{tr}(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$

$$\mathrm{tr}(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij}\, \mathrm{tr}(\underline{e}_i \otimes \underline{e}_j) = A_{ij}\, \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} = A_{ii}$$

Properties of trace:

$$\mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Decomposition: $\boxed{\underline{\underline{A}} = \alpha \underline{\underline{I}} + \mathrm{dev}(\underline{\underline{A}})}$

Spherical tensor: $\alpha \underline{\underline{I}}$ where $\alpha = \frac{1}{3}\mathrm{tr}(\underline{\underline{A}})$

Deviatoric tensor: $\mathrm{dev}(\underline{\underline{A}}) = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\mathrm{tr}(\mathrm{dev}(\underline{\underline{A}})) = 0$$

Tensor scalar product (contraction)

analogous to scalar product of vectors

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}$$

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \dots$$

$$A_{21} B_{21} + \dots$$

Show $\cancel{\underline{\underline{A}} : \underline{\underline{B}}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}$

on HW2 $\underline{\underline{A}}^T \underline{\underline{B}} = A_{ji} B_{jl} \, \underline{e}_i \otimes \underline{e}_l$ $\qquad \underline{e}_j = \delta_{ij} \underline{e}_i$

$\underline{e}$

$$tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ji} B_{jl} \, tr(\underbrace{\underline{e}_i \otimes \underline{e}_l}) =$$

$$\rightarrow \quad = A_{ji} B_{jl} \, \underbrace{\underline{e}_i \cdot \underline{e}_l} = A_{ji} B_{ji} \quad \checkmark$$

$$B_{jl} \delta_{il} = B_{ji}$$

Properties: 1) $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{B}} : \underline{\underline{A}}$

2) $(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$

Symmetry follows from prop. of trace

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underbrace{\underline{\underline{A}}^T \underline{\underline{B}}}_{C}) = tr((\underline{\underline{A}}^T \underline{\underline{B}})^T) = tr(\underline{\underline{B}}^T \underline{\underline{A}}) = \underline{\underline{B}} : \underline{\underline{A}}$$

Second prop: $[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$

$$(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = a_i b_j \, c_i d_j = a_i c_i \, b_j d_j = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

$$A : B = A_{ij} B_{ij}$$

A common norm for tensors is

$$|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = \sqrt{A_{ij} A_{ij}} \geq 0$$

Application: used to compute the work done during deformation.

$\Rightarrow$ shear heating in glaciology

Determinant & Inverse

$$\det(\underline{\underline{A}}) = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \epsilon_{ijk} \underbrace{[\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}}_{\text{columns of } \underline{\underline{A}}}$$

$[A]_{i1}$ $[A]_{j2}$

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

$$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}}) \qquad \underline{\underline{A}} \text{ is } n \times n$$

→ determinants are important for volume changes.

$\underline{\underline{A}}$ is singular if $\det(\underline{\underline{A}}) = 0$

If $\det(\underline{\underline{A}}) \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\boxed{\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Properties: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

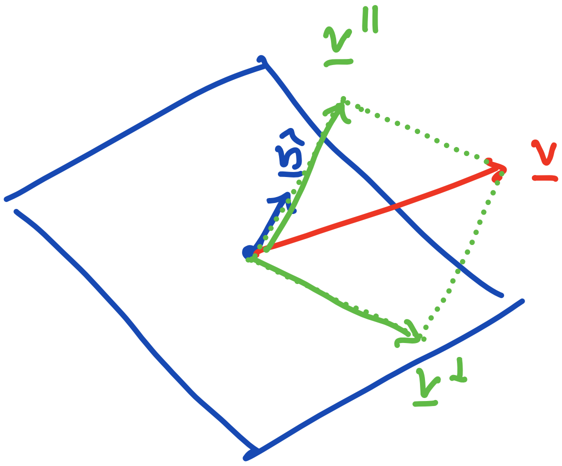
$$(\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1}$$

$$(\alpha \underline{\underline{A}})^{-1} = \frac{1}{\alpha}\underline{\underline{A}}^{-1}$$

$$\det(\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}})^{-1} = \frac{1}{\det(\underline{\underline{A}})}$$

Projection & Reflections

commonly want to partition forces on a surface



$$\underline{v} = \underline{v}^{\parallel} + \underline{v}^{\perp}$$

$$\underline{v}^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}} \leftarrow \text{dyadic}$$

$$\underline{v}^{\perp} = \underline{v} - \underline{v}$$

dyadic product: $(\underline{a} \otimes \underline{b})\, \underline{v} = (\underline{b} \cdot \underline{v})\, \underline{a}$

$$\underline{v}^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}} = (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{\underline{P}}_n^{\parallel}\, \underline{v}$$

$$\underline{v}^{\perp} = \underline{v} - \underline{v}^{\parallel} = \underline{\underline{I}}\,\underline{v} - (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = (\underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v}$$

$$= \underline{\underline{P}}_n^{\perp}\, \underline{v}$$

Parallel and perp. projection tensors

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \hat{\underline{n}} \otimes \hat{\underline{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}}}$$

PSo

Properties of projection matrices:

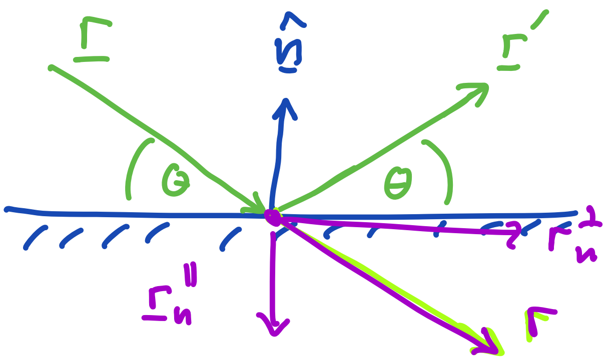
$$\underline{\underline{P}}_n = \underline{\underline{P}}_n^T$$

$$\underline{\underline{P}}_n^2 = \underline{\underline{P}}_n$$

$$\underline{\underline{P}}_n^{\parallel} + \underline{\underline{P}}_n^{\perp} = \underline{\underline{I}}$$

$$\underline{\underline{P}}^{\parallel}\,\underline{\underline{P}}^{\perp} = \underline{\underline{0}}$$

Reflections



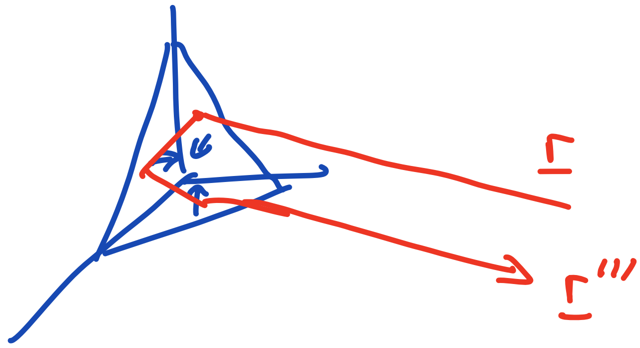
incoming: $\underline{r} = \underline{r}_n^{\parallel} + \underline{r}_n^{\perp}$

outgoing: $\underline{r}' = -\underline{r}_n^{\parallel} + \underline{r}_n^{\perp}$

$$\underline{r}' = \left(-\underline{\underline{P}}_n^{\parallel} + \underline{\underline{P}}_n^{\perp}\right)\underline{r} = \Big(\underbrace{\underline{\underline{I}} - \underline{\hat{n}}\otimes\underline{\hat{n}}}_{\underline{\underline{P}}^{\perp}} - \underline{\hat{n}}\otimes\underline{\hat{n}}\Big)\underline{r}$$

$$\underline{r}' = \underbrace{\left(\underline{\underline{I}} - 2\,\underline{\hat{n}}\otimes\underline{\hat{n}}\right)}_{\underline{\underline{R}}_n}\underline{r}$$

On HW: Corner reflector



$$\underline{r}''' = -\underline{r}$$

$$\underline{r}''' = \underline{\underline{R}}_{n_1} \underline{\underline{R}}_{n_2} \underline{\underline{R}}_{3} \underline{r}$$