Lecture 4: Cauchy Stress Tensor

Logistics: – HW1 is due

– thanks to Afzal!

– HW2 is posted

Last time: – Tensor algebra in index notation

$$H_{il} = S_{ij} T_{jl}$$

– Transpose: $\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}$ $\quad S^T_{ij} = S_{ji}$

⇒ Symmetric Skew decomposition

– Trace : $tr(\underline{\underline{A}}) = A_{ii}$

⇒ Spherical - deviatoric decomposition

– Tensor scalar product: $\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}$

– Determinant & Inverse

– Projection & Reflection

Today: Cauchy stress tensor

⇒ continuum mass & force concepts

• Body & surface forces

- Traction field & Cauchy's postulate
- Stress tensor & Cauchy's theorem

Continuum Mass & Force Concepts

Continuum body → spatially distributed
infinitely divisible

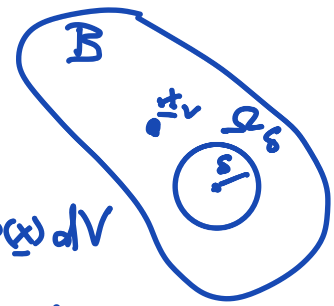
this is o.k. at length scales much larger than the interatomic spacing

Mass density

Mass is a phys. property of matter that quantifies its resistance to acceleration when a force is applied.

Assume that mass is continuously distributed throughout a body B

any ~~po~~ subset of B with pos. volume has a positive mass



$$V_\Omega = \int dV \qquad m_\Omega = \int \rho(\underline{x})\, dV$$

$\rho(\underline{x})$ is the mass density field at any point $\underline{x}$ ρ is defined as

$$\boxed{\rho(\underline{x}) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}}$$

The center of volume of B

$$\underline{x}_v = \frac{1}{V_B} \int_B \underline{x}\, dV$$

the center of mass

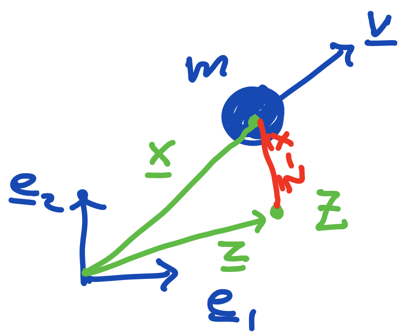
$$\underline{x}_m = \frac{1}{m_B} \int \rho(\underline{x})\, \underline{x}\, dV$$

Short review of force & moment

Object with mass m and velocity $\underline{v}$ has momentum:

Linear momentum: $\underline{L} = m\underline{v}$

Angular momentum: $\underline{j}_z = (\underline{x} - \underline{z}) \times \underline{L}$



always relative to a point

Newtons 1st law: "Principle of inertia"

In a fixed frame of reference every object preserves its state of motion unless it is acted upon by a force or a torque.

Force: $\underline{f} = \frac{d\underline{L}}{dt} = \dot{\underline{L}} = m\dot{\underline{v}} = m\underline{a}$

$\rightarrow$ Newtons 2nd law

Torque: $\underline{\tau} = \frac{d\underline{j}}{dt} = m(\underline{x} - \underline{z}) \times \underline{a}$

Body Force

any force <u>not</u> due to physical contact is a body force: Common body forces arise from gravitational or electro magnetic fields.

If $\underline{b}(\underline{x})$ is body force field units $\frac{\text{force}}{\text{volume}}$

$$\frac{F}{V} = \frac{am}{V} = \frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}$$

the resultant force on body is

$$\underline{r}_b = \int_B \underline{b}(\underline{x})\, dV$$

the resultant torque on a body about point z is given by

$$\underline{\tau}_b = \int_B (\underline{x} - \underline{z}) \times \underline{b}(\underline{x})\, dV$$

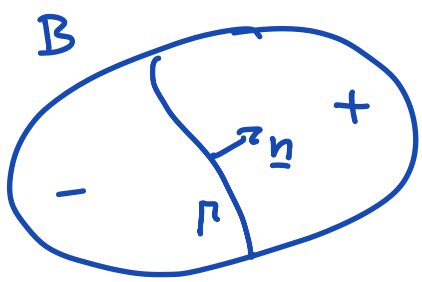
Example: gravitational body force

$$\underline{b}_g = \rho g \qquad \frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}$$

Surface forces

arise due to physical contact between bodies. Forces along the boundaries are external forces. Forces along imaginary surfaces within a body are internal forces.

Traction field



arbitrary surface Γ with normal $\underline{n}(\underline{x})$

The force per unit area exerted by material on pos. side upon the material on neg. side is given by the traction field $\underline{t}_n$

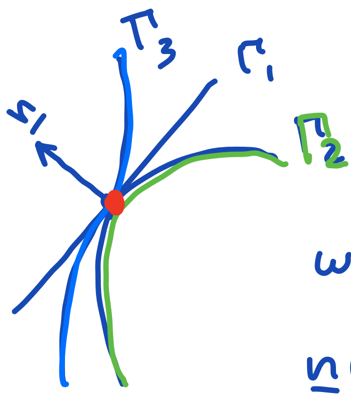
the resultant force due to a traction field on Γ is

$$\boxed{\underline{r}_s[\Gamma] = \int_\Gamma \underline{t}_n(\underline{x})\, dA}$$

the resultant torque about point z is

$$\boxed{\underline{\tau}_s[\Gamma] = \int_\Gamma (\underline{x}-\underline{z}) \times \underline{t}_n(\underline{x})\, dA}$$

Cauchy's postulate



The traction field $\underline{t}_n$ on Γ in B depends only pointwise on the unit normal field $\underline{n}(\underline{x})$. In particular there is a traction function such that

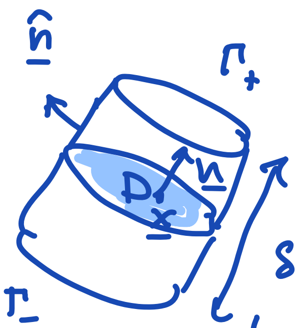
$$\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}),\underline{x})}$$

assumes that $\underline{t}_n$ is independent of $\nabla\underline{n}$ and hence the curvature of ~~the~~ Γ.

Law of Action and Reaction

If $\underline{t}(\underline{n},\underline{x})$ is continous and bouded

then $$\boxed{\underline{t}(-\underline{n},\underline{x}) = -\underline{t}(\underline{n},x)}$$ for all $\underline{n}$ and $\underline{x}$



To show this consider
disk D with ~~fi~~ radius around
$\underline{x}$. Let Ω_δ be cylinder
with center $\underline{x}$ and axis $\underline{n}$ and height $\delta > 0$.
End faces Γ_+ Γ_- and mantle Γ_δ

Note: $\hat{\underline{n}} = \underline{n}$ on Γ_+

$\hat{\underline{n}} = -n$ on Γ_-

As $\delta \to 0$ $\quad \Gamma_+$ and $\Gamma_- \to D$

$\Gamma_\delta \to 0$

Entire surface area: $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_+ \cup \Gamma_-$

$$\lim_{\delta \to 0} \left[\int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}}(y), \underline{y})\, dA + \int_{\Gamma_+} \underline{t}(\underline{n}, \underline{y})\, dA + \int_{\Gamma_-} \underline{t}(-\underline{n}, \underline{y})\, dA \right] = 0$$

the first term vanishes because $\Gamma_\delta \to 0$

As $\delta \to 0$ $\quad \Gamma_+$ & $\Gamma_- \to D$

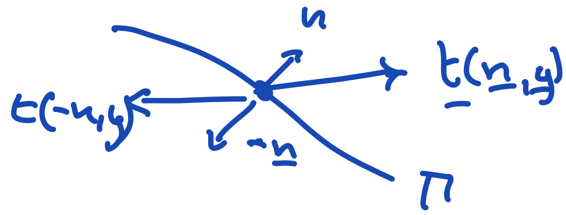
$$\int_D \underline{t}(\underline{n}, \underline{y}) + \underline{t}(-\underline{n}, y)\, dA = 0$$

Because radius of D is arbitrary the interior of the integrand must be zero

$$\underline{t}(\underline{n},\underline{y}) + \underline{t}(-\underline{n},\underline{y}) = 0$$

$$\underline{t}(-\underline{n},\underline{y}) = -\underline{t}(\underline{n},\underline{y}) \checkmark$$

→ action and reaction



The stress tensor

Cauchy's theorem

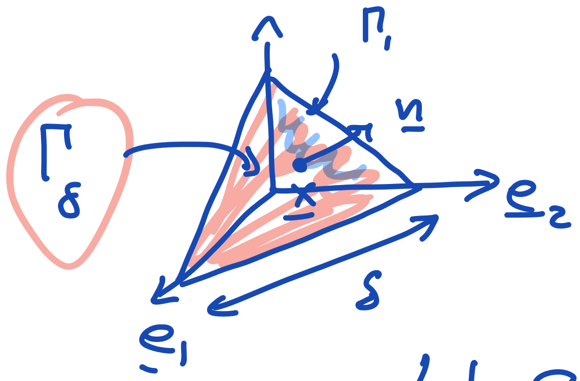
Let $\underline{t}(\underline{n},\underline{x})$ be a traction field for body $\mathcal{B}$ that satisfies Cauchy's postulate. Then $\underline{t}(\underline{n},\underline{x})$ is linear in $\underline{n}$, then there exists a second order tensor field $\underline{\underline{\sigma}}(\underline{x}) \in \mathcal{V}^2$ such that

$$\boxed{\underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$$

called the Cauchy stress.

frame $\{\underline{e}_i\}$ point $\underline{x} \in \mathcal{B}$

e_3



Cauchy stress tetrahedron

$\underline{n} \cdot \underline{e}_i > 0$

Let Ω_δ be the tetrahedron bounded by Γ_δ and $\Gamma_1, \Gamma_2, \Gamma_3$ where Γ_i is triangle with $\underline{e}_i$ as normal

$$\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} t(\underline{n}(y), \underline{y})\, dA = 0$$

$$\frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{x})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j, \underline{x})\, dA \right] = 0$$

you can show $A_{\Gamma_j} = n_j A_{\Gamma_\delta}$

$$\Rightarrow A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=i}^{3} \Gamma_j = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substituting

$$\lim_{\delta \to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \left(\underline{t}(\underline{n}, \underline{y}) + \sum_{j=1}^{3} \underline{t}(\underline{e}_j, \underline{x})\, n_j \right) dA$$

As $\delta \to 0$ the Area Γ_δ shrinks to $\underline{x}$ so that by mean value Thm for integrals the limit is given by integrand

$$t(\underline{n}, \underline{x}) + \sum_{j=1}^{3} t(-\underline{e}_j, \underline{x})\, n_j = 0$$

use law of action & reaction

$$\underline{t}(\underline{n}, \underline{x}) = \sum_{j=1}^{3} \underline{t}(\underline{e}_j, \underline{x})\, n_j$$

with summation convention

$$\underline{t}(\underline{n}, \underline{x}) = \underline{t}(\underline{e}_j, \underline{x})\, n_j$$

what does the r.h.s mean?

Consider $(\underline{a} \otimes \underline{b})\underline{c} = (\underline{b} \cdot \underline{c})\, \underline{a}$

$$(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j)\, \underline{n} = (\underline{e}_j \cdot \underline{n})\, \underline{t}(\underline{e}_j, \underline{x})$$

$$= (\underline{e}_j \cdot n_i \underline{e}_i)\, \underline{t}(\underline{e}_j, \underline{x})$$

$$= n_i (\underline{e}_j \cdot \underline{e}_i)$$

$$n_i \quad \delta_{ij} \quad \underline{t}(\underline{e}_j, \underline{x})$$

$$= n_j\, \underline{t}(\underline{e}_j, \underline{x})$$

~~so~~ setting these equal

$$\underline{t}(\underline{n},\underline{x}) = \underbrace{\left(\underline{t}(\underline{e}_j,\underline{x})\otimes \underline{e}_j\right)}_{\underline{\underline{\sigma}}(\underline{x})} \underline{n} = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}$$

write $\quad \underline{t}(\underline{e}_j, x) = t_i(\underline{e}_j,\underline{x})\,\underline{e}_i$

$$\underline{\underline{\sigma}} = \underline{t}(\underline{e}_j,\underline{x})\otimes\underline{e}_j = \underbrace{t_i(\underline{e}_j,x)}_{\sigma_{ij}}\ \underbrace{\underline{e}_i\otimes\underline{e}_j}_{\text{tensor basis}}$$

The definition of the Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\,\underline{e}_i\otimes\underline{e}_j \quad \text{with} \quad \sigma_{ij} = t_i(\underline{e}_j,\underline{x})}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane