Lecture 5: Stress tensor applications

Logistics: - HW 1 has been graded

- HW2 is due Th

Last time: - body & surface forces

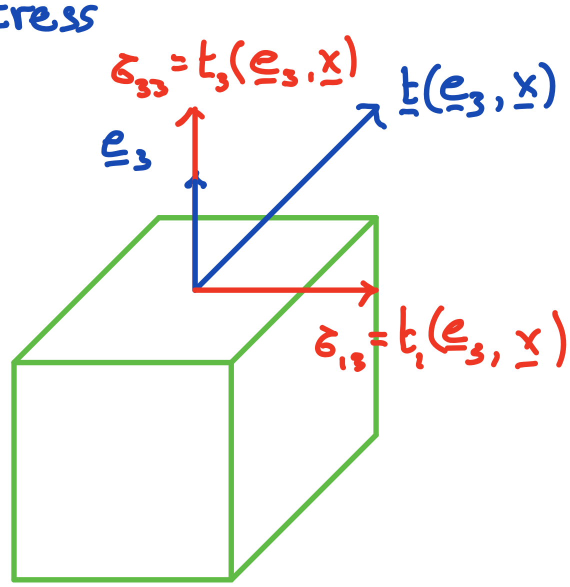
- traction field : $\underline{t}_n = \underline{t}(\underline{n}(\underline{x}), \underline{x})$
- Action & reaction: $\underline{t}(-\underline{n}, \underline{x}) = -\underline{t}(\underline{n}, \underline{x})$
- Cauchy's theorem: $\boxed{\underline{t}_n = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$

$\underline{\underline{\sigma}}$ = Cauchy stress

- $\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$\sigma_{ij} = t_i(\underline{e}_j, \underline{x})$

i-th component of traction on the j-th coordinate plane.

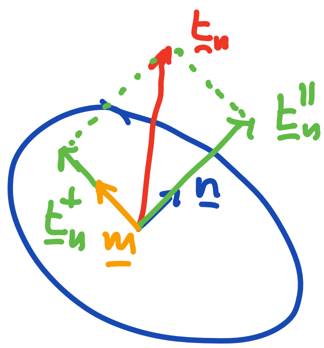


Today: - Normal and shear stress

- Simple states of stress

- Archimedes principle
- Thrust sheet

Normal and Shear stress



Two projection matrices:

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n} \qquad \underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

$\underline{m}$ = tangential vector

normal stress: $\underline{t}^{\parallel} = \underline{\underline{P}}^{\parallel} \underline{t}_n = (n \otimes n)\, \underline{t}_n$

$$= \underbrace{(\underline{n} \cdot \underline{t}_n)}_{\sigma_n}\, \underline{n} = \sigma_n \underline{n}$$

shear stress: $\underline{t}^{\perp} = \underline{\underline{P}}^{\perp} \underline{t}_n = (\underline{m} \otimes \underline{m}) = (\underline{m} \cdot \underline{t}_n)\, \underline{m}$

$$= \tau\, \underline{m}$$

In index notation:

$$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} \Rightarrow \quad \sigma_n = n_i \sigma_{ij} n_j$$

$$\tau = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \underline{\underline{\sigma}}\, \underline{n} \Rightarrow \quad \tau = m_i \sigma_{ij} n_j$$

If $\sigma_n > 0 \Rightarrow$ tensile stress

$\sigma_n < 0 \quad \Rightarrow \quad$ compressive stress

Simple states of stress

I) Hydrostatic stress

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

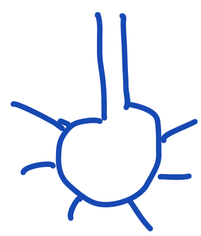
$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{\underline{I}}\,\underline{n} = -p\,\underline{n}$$

normal stress vector:

$$\underline{t}_n^{\parallel} = \underline{\underline{P}}_n^{\parallel}\,\underline{t}_n = (\underline{n}\otimes\underline{n})(-p\,\underline{n}) = -p\,(\underline{n}\otimes\underline{n})\,\underline{n}$$

$$= -p\,(\underline{n}\cdot\underline{n})\,\underline{n} = -p\,\underline{n}$$

normal stress: $\sigma_n = -p$



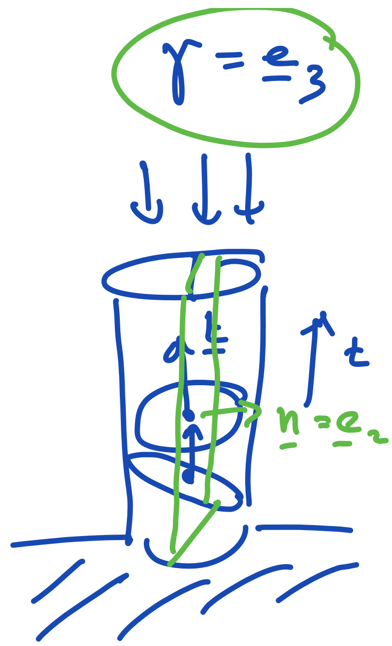
shear stress: $\underline{t}_n = \underline{t}_n^{\parallel} + \underline{t}_n^{\perp}$

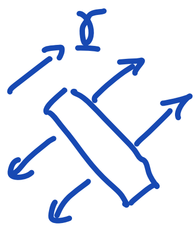
$$\underline{t}_n^{\perp} = \underline{t}_n - \underline{t}_n^{\parallel} = \underline{0} \quad \Rightarrow \quad \tau = 0$$

$\Rightarrow$ no shear stress on any plane

II, Uniaxial stress

$$\boxed{\underline{\underline{\sigma}} = \sigma\, \underline{\gamma} \otimes \underline{\gamma}} \qquad |\underline{\gamma}| = 1$$





traction

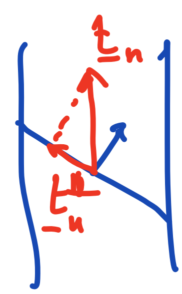
$$\begin{aligned} \underline{t}_n &= \underline{\underline{\sigma}}\, \underline{n} \\ &= \sigma (\underline{\gamma} \otimes \underline{\gamma})\, \underline{n} \\ &= \sigma (\underline{\gamma} \cdot \underline{n})\, \underline{\gamma} \end{aligned}$$

$\Rightarrow$ traction is always parallel to $\underline{\gamma}$

and vanishes on surfaces perp. to $\underline{\gamma}$

$\sigma < 0$: pure compression

$\sigma > 0$: pure tension



III Pure shear stress

two directions: $\underline{\gamma}\cdot\underline{\eta}=0$ perpendicular.

$$\boxed{\underline{\underline{\sigma}} = \tau\,(\underline{\gamma}\otimes\underline{\eta}+\underline{\eta}\otimes\underline{\gamma})}$$

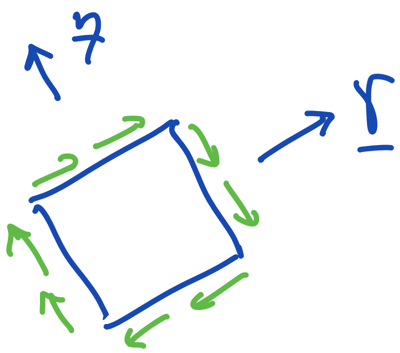
traction: $\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \tau\,(\underline{\gamma}\otimes\underline{\eta})\,\underline{n} + \tau\,(\underline{\eta}\otimes\underline{\gamma})\,\underline{n}$

$$= \tau\,(\underline{\eta}\cdot\underline{n})\,\underline{\gamma} + \tau\,(\underline{\gamma}\cdot\underline{n})\,\underline{\eta}$$

traction on $\underline{n}=\underline{\eta}$: $\underline{t}_n = \tau\,\cancel{(\underline{\eta}\cdot\underline{n})}^{1}\,\underline{\gamma} + \tau\,\cancel{(\underline{\gamma}\cdot\underline{n})}^{0}\,n$

$$\underline{t}_n = \tau\,\underline{\gamma}$$

traction on $\underline{n}=\underline{\gamma}$: $\underline{t}_n = \tau\,\underline{\eta}$



IV Plane stress

If there exist a pair of orthonormal vectors $\underline{\gamma}$ and $\underline{\eta}$ such that matrix representation of $\underline{\underline{\sigma}}$ in frame $\{\underline{x}, \underline{y}, \underline{x} \times \underline{y}\}$ is of form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then a state of plane stress exists.

$\Rightarrow$ 2D problem

Spherical & deviatoric stress tensors

Cauchy stress can be decomposed

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_S + \underline{\underline{\sigma}}_D$$

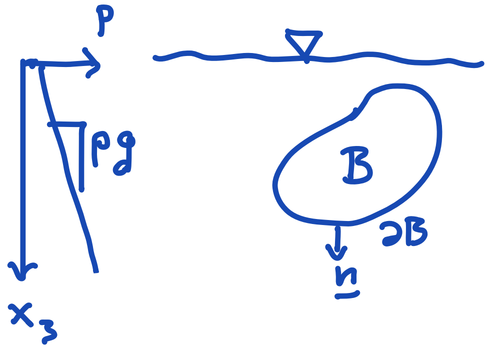
spherical stress tensor: $\underline{\underline{\sigma}}_S = -p\underline{\underline{I}}$ $\quad p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})$

deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} + p\underline{\underline{I}}$

The pressure $p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

~~pre~~ can be interpreted as mean normal stress

Example: Archimedes' principle



Any submerged object in fluid is buoyed up by a force equal to the weight of the displaced fluid.

Q: Is the buoyancy force a body or a surface force?

Hydrostatic pressure acts on boundary of the object ⇒ external surface force

Buoyancy force is resultant surface force

$$\boxed{\underline{F}_B = -W\underline{e}_3 = -\rho g V_B\, \underline{e}_3}$$ ρ = water density

Hydrostatic pressure: $p = \rho g x_3$

$$\nabla p = \rho g \underline{e}_3$$

Hydrostatic traction on ∂B: $\underline{t}_n = -p\,\underline{n}$

Resultant surface force:

$$\underline{r}_s = \int_{\partial B} \underline{t}_n \, dA = - \int_{\partial B} p \, \underline{n} \, dA$$

need to convert to volume integral

$\Rightarrow$ Divergence Thm: $\displaystyle\int_{\Omega} \nabla \cdot \underline{f} \, dV = \int_{\partial\Omega} \underline{f} \cdot \underline{n} \, dA$

to convert multipy by arbitrary but constant vector $\underline{c} \neq \underline{c}(\underline{x})$

$$\underline{c} \cdot \underline{r}_s = - \underline{c} \int_{\partial B} p \, \underline{n} \, dA = - \int_{\partial B} \underline{c} \cdot (p \underline{n}) \, dA = - \int_{\partial B} \underbrace{(p \underline{c})}_{\underline{f}} \cdot \underline{n} \, dA$$

apply div. theorem.

$$\underline{c} \cdot \underline{r}_s = - \int_B \nabla \cdot (p \, \underline{c}) \, dV = - \int_B \underline{c} \cdot \nabla p + p \, \cancelto{0}{\nabla \cdot \underline{c}} \, dV$$

$$\underline{c} \cdot \underline{r}_s = - \int_B \underline{c} \cdot \nabla p \, dV = - \underline{c} \cdot \int_B \nabla p \, dV$$

because $\underline{c}$ is arbitrary

$$\underline{F}_S = -\int_B \nabla p \, dV \qquad \nabla p = \rho g \, \underline{e}_3$$

$$= -\rho g \, \underline{e}_3 \underbrace{\int_B dV}_{V_B} = -\rho g V_B \, \underline{e}_3 \quad \checkmark$$