Lecture 6: Change in basis & eigen problem

Logistics: - HW1 is graded
- HW2 is due
- HW3 will be posted

Last time: - Applications of stress tensor
- Normal & shear stress
- Simple states of stress
  - hydrostatic, uniaxial, pure shear
- Example of Archimedes principle

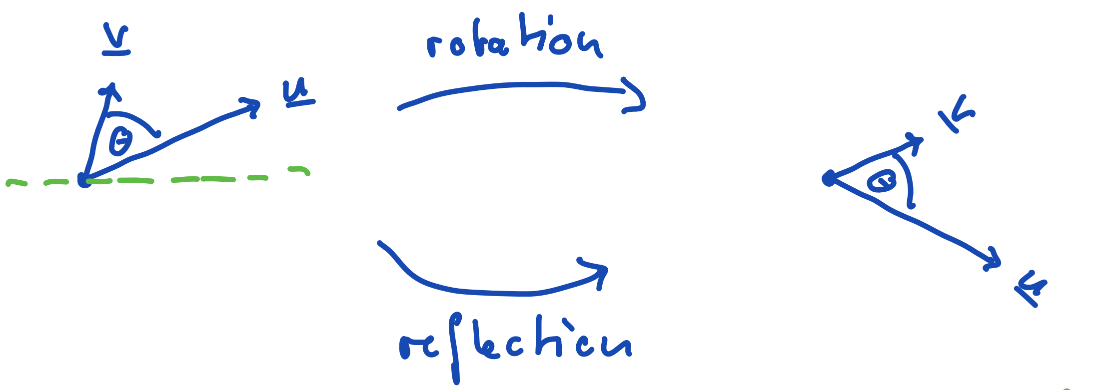
Today: More tensor algebra
- orthogonal tensor
- change in basis/representation
- eigen problem
- spectral decomposition
- principal invariants

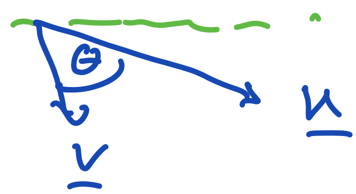
Orthogonal tensors

$\underline{\underline{Q}}$ is orthogonal tensor if

$$\boxed{\underline{\underline{Q}}\,\underline{u} \cdot \underline{\underline{Q}}\,\underline{v} = \underline{u} \cdot \underline{v}} = |\underline{u}|\,|\underline{v}| \overset{\cos}{\cancel{\sin}}\,\theta$$

$\Rightarrow$ preserves length $\underline{u}$ & $\underline{v}$ and angle θ





Properties:

$$\boxed{\begin{aligned} \underline{\underline{Q}}^T &= \underline{\underline{Q}}^{-1} \\ \underline{\underline{Q}}^T \underline{\underline{Q}} &= \underline{\underline{I}} \\ \det(\underline{\underline{Q}}) &= \pm 1 \end{aligned}}$$

$$\det(\underline{\underline{I}}) = 1 \Rightarrow \det(\underline{\underline{Q}}^T \underline{\underline{Q}}) = \det(\underline{\underline{Q}}^T)\det(\underline{\underline{Q}})$$

$$= \det(\underline{\underline{Q}})^2 = 1$$

If $\det(\underline{\underline{Q}}) = 1 \Rightarrow$ rotation

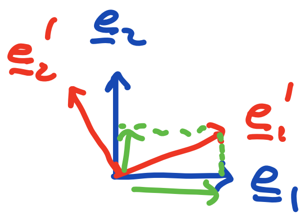
$\det(\underline{\underline{Q}}) = -1 \Rightarrow$ reflection

Change in basis

Both $\underline{v} \in \mathcal{V}$ and $\underline{\underline{S}} \in \mathcal{V}^2$ are invariant upon change of basis, but their representations $[\underline{v}]$ and $[\underline{\underline{S}}]$ change.

Consider two frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

In 2D



Representation of $\underline{e}'_j$ in $\{\underline{e}_i\}$

$$\underline{e}'_j = (\underline{e}'_j \cdot \underline{e}_1)\,\underline{e}_1 + (\underline{e}'_j \cdot \underline{e}_2)\,\underline{e}_2 + (\underline{e}'_j \cdot \underline{e}_3)\,\underline{e}_3$$

$$= (\underline{e}'_j \cdot \underline{e}_i)\,\underline{e}_i = (\underbrace{\underline{e}_i \cdot \underline{e}'_j}_{A_{ij}})\,\underline{e}_i$$

$$\boxed{\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j}$$

where $\underline{\underline{A}}$ is the <u>change of basis</u> tensor

Note $A_{ij}\, \underline{e}_i$ is transpose

Similarly we can expresse $\underline{e}_i$ in $\{\underline{e}'_k\}$

$$\underline{e}_i = (\underline{e}_i \cdot \underline{e}'_k)\,\underline{e}'_k = A_{ik}\,\underline{e}'_k \qquad \underline{e}'_k = A_{\ell k}\,\underline{e}_\ell$$

We have : $\underline{e}'_j = A_{ij}\,\underline{e}_i$ $\qquad$ $\underline{e}_i = A_{ik}\,\underline{e}'_k$

$$\underline{e}'_j = \underbrace{A_{ij}A_{ik}}_{\delta_{jk}}\,\underline{e}'_k$$

from frame identity

$$A_{ij}A_{ik} = \delta_{jk}$$

$$\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{e}_i = \underbrace{A_{ik}A_{\ell k}}_{\delta_{i\ell}}\,\underline{e}_\ell$$

$$\underline{e}_i = \delta_{iL}\,\underline{e}_L$$ frame identity

$$A_{ik}A_{\ell k} = \delta_{iL}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}$$

$\Rightarrow$ $\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$ $\Rightarrow$ $\underline{\underline{A}}$ is orthogonal!

if both $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ are right handed

$\Rightarrow$ $\underline{\underline{A}}$ must be a rotation $\det(\underline{\underline{A}}) = 1$

Change in representation

Consider $\underline{v}$ and $\underline{\underline{S}}$

with $[\underline{v}]$ $[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$

$[\underline{v}]'$ $[\underline{\underline{S}}]'$ in $\{\underline{e}'_j\}$

so that $[\underline{v}] \neq [\underline{v}]'$ $[\underline{\underline{S}}] \neq [\underline{\underline{S}}]'$

then $\boxed{[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \qquad [v'] = [\underline{\underline{A}}]^T[\underline{v}]}$

$\underline{v} = v_i \underline{e}_i = v'_j \underline{e}'_j$ where $\underline{e}'_j = A_{ij} \underline{e}_i$

$v_i \underline{e}_i = v'_j A_{ij} \underline{e}_i$

$\Rightarrow \quad v_i = A_{ij} v'_j$

Similarly

$$\boxed{\begin{aligned} [\underline{\underline{S}}] &= [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \\ [\underline{\underline{S}}]' &= [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}] \end{aligned}}$$

Change in basis is rotation $\Rightarrow \underline{\underline{A}}$ is orthogonal

$A_{ij} = \underline{e}_i \cdot \underline{e}'_j$

Invariance of the trace

$[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$ $\qquad$ $[\underline{\underline{S}}]'$ in $\{\underline{e}_j'\}$

$$\boxed{tr[\underline{\underline{S}}] = tr[\underline{\underline{S}}]'}$$

consider

$[\underline{\underline{S}}] = [\underline{A}][\underline{\underline{S}}]'[\underline{A}]^T$ or $[\underline{\underline{S}}]_{ij} = [A]_{ik}[S]'_{kl}[A]_{jl}$

$$tr[\underline{\underline{S}}] = [\underline{\underline{S}}]_{ii} = [A]_{ik}[S]'_{kl}[A]_{il}$$

$$= \underbrace{[\underline{\underline{A}}]_{ik}[\underline{\underline{A}}]_{il}}_{\delta_{kl}}[\underline{\underline{S}}]'_{kl}$$

$$\delta_{kl}[\underline{S}]'_{kl} = [\underline{\underline{S}}]'_{kk} = tr[\underline{\underline{S}}]'$$

Invariance of determinant

$det[\underline{\underline{S}}] = det[\underline{\underline{S}}]' \quad \Rightarrow$ HW 3

Eigenvalues and Eigenvectors of tensors

By the eigenpair of $\underline{\underline{S}} \in \mathcal{V}^2$ we mean the scalar λ and vector $\underline{v}$ such that

$$\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v} \qquad (\underline{\underline{S}} - \lambda \underline{\underline{I}})\,\underline{v} = \underline{0}$$

λ = eigenvalue $\qquad \underline{v}$ = eigenvector

λ's are roots of characteristic polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p we have one or more $\underline{v}_p$ satisfying $\qquad (\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = 0$

Here we are mostly concerned with symmetric tensors.

Eigenproblem for symmetric tensors

1) All λ_p's are real

2) All λ_p's are positive ($\underline{\underline{S}}$ sym. pos. def)

3) All $\underline{v}_p$' corresponding to distinct λ_p are <u>orthogonal</u>

If $\underline{\underline{S}}$ is sym. pos. def. (spd)

if $\boxed{\underline{v} \cdot \underline{\underline{S}}\,\underline{v} > 0}$ for all $\underline{v} \in \mathcal{V}$ $\underline{v} \neq \underline{0}$

by def of eigenpair $\underline{\underline{S}}\,\underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot (\lambda \underline{v}) > 0$$

$$\lambda \; \underline{v} \cdot \underline{v} = \lambda |\underline{v}|^2 > 0 \Rightarrow \lambda > 0$$

To establish orthogonality of eigenvectors two distict eigenpairs $(\lambda, \underline{v})$ and $(\omega, \underline{u})$ so that $\lambda \neq \omega$

$$\underline{\underline{S}}\,\underline{v} = \lambda \underline{v} \qquad \text{and} \qquad \underline{\underline{S}}\,\underline{u} = \omega \underline{u}$$

Consider: $\lambda(\underline{v} \cdot \underline{u}) = (\lambda \underline{v} \cdot \underline{u}) = (\underline{\underline{S}}\,\underline{v}) \cdot \underline{u}$

$$= \underline{v} \cdot \underline{\underline{S}}^T \underline{u} \overset{\underline{\underline{S}} = \underline{\underline{S}}^T}{=} \underline{v} \cdot \underline{\underline{S}}\,\underline{u} = \underline{v} \cdot (\omega \underline{u})$$

$$= \omega (\underline{v} \cdot \underline{u})$$

$$\Rightarrow \quad \lambda(\underline{v}\cdot\underline{u}) = \omega(\underline{v}\cdot\underline{u}) \qquad \lambda \neq \omega = 0$$

$$\Rightarrow \quad \underline{v}\cdot\underline{u} = 0 \quad \Rightarrow \quad \underline{v} \perp \underline{u}$$

Spectral decomposition

If $\underline{\underline{S}} = \underline{\underline{S}}^T$ there exists a frame $\{\underline{v}_i\}$ consisting of the eigenvectors of $\underline{\underline{S}}$ so that

$$\boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \; \underline{v}_i \otimes \underline{v}_i} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

Consider $\quad \underline{\underline{I}} = \underline{v}_i \otimes \underline{v}_i$

$$\underline{\underline{S}}\,\underline{\underline{I}} = \underline{\underline{S}}(\underline{v}_i \otimes \underline{v}_i) \overset{!\ \text{HW}}{=} (\underline{\underline{S}}\,\underline{v}_i)\otimes\underline{v}_i = \sum_{i=1}^{3} (\lambda_i \underline{v}_i)\otimes\underline{v}_i$$

$$\overset{!}{=} \sum_{i=1}^{3} \lambda_i (\underline{v}_i \otimes \underline{v}_i)$$

$\Rightarrow$ Spectral decomposition diagonalizes the tensor

The principal invariants of $\underline{\underline{S}} = \underline{\underline{S}}^T$ are

$$\boxed{\begin{aligned} I_1(\underline{\underline{S}}) &= tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2(\underline{\underline{S}}) &= \tfrac{1}{2}\left((tr(S))^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ I_3(\underline{\underline{S}}) &= det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3 \end{aligned}}$$

These 3 scalars are frame invariant

Set of invariants $I_S = \{I_i(\underline{\underline{S}})\}$

Rewrite char. polynomial with invariants

$$det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = -\lambda^3 + I_1(\underline{\underline{S}})\lambda^2 - I_2(\underline{\underline{S}})\lambda + I_3(\underline{\underline{S}}) = 0$$

<u>Cayley - Hamilton Thm</u>

every tensor satisfies its own char. polynomial.

$$\boxed{-\underline{\underline{S}}^3 + I_1(\underline{\underline{S}})\underline{\underline{S}}^2 - I_2(\underline{\underline{S}})\underline{\underline{S}} + I_3(\underline{\underline{S}}) = 0}$$

$$\underline{\underline{S}}\,\underline{\underline{S}}\,\underline{v} = \underline{\underline{S}}(\lambda \underline{v}) = \lambda\,\underline{\underline{S}}\,\underline{v} = \lambda^2 \underline{v}$$

$$\boxed{\underline{\underline{S}}^{\alpha}\underline{v} = \lambda^{\alpha}\underline{v}}$$

multiply char. poly. by $\underline{v}$

$$-\lambda^3\underline{v} + I_1(\underline{\underline{S}})\lambda^2\underline{v} - I_2(\underline{\underline{S}})\lambda\underline{v} + I_3(\underline{\underline{S}})\underline{v} = 0$$

$$-\underline{\underline{S}}^3\underline{v} + I_1(\underline{\underline{S}})\underline{\underline{S}}^2\underline{v} - I_2(\underline{\underline{S}})\underline{\underline{S}}\underline{v} + I_3(\underline{\underline{S}})\underline{v} = 0$$

$\Rightarrow$ Caley-Hamilton