Lecture 7: Maximum Normal & Shear stresses

Logistics: - HW 3 due Thursday

Last time: - Orthogonal tensors $\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$ $\det(\underline{\underline{Q}}) = \pm 1$

rotation or reflection

- Change in basis: $\{\underline{e}_i\}$ $\{\underline{e}_i'\}$;

$\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$ $\quad A_{ij} = \underline{e}_i \cdot \underline{e}_j'$

$\Rightarrow \underline{\underline{A}}$ rotation

$[\underline{v}] = [\underline{\underline{A}}][\underline{v}]'$ $\quad [\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T$

- Invariance of $\mathrm{tr}[\underline{\underline{S}}]$ & $\det[\underline{\underline{S}}]$

- Eigen problem: $\boxed{\underline{\underline{S}}\,\underline{v} = \lambda\, \underline{v}}$

$\underline{\underline{S}}$ is sym. pos. def.

$\Rightarrow \underline{v}$'s are orthogonal

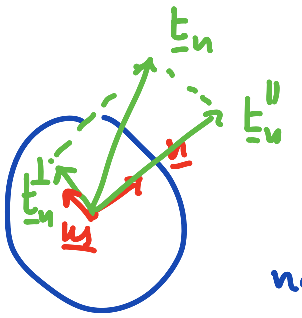
- Spectral decomposition

$$\boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i\, \underline{v}_i \otimes \underline{v}_i}$$

- Principle invariants

Today: Max Normal & Shear stresses

Normal and Shear Stress



$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n}$$

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n} \qquad \underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

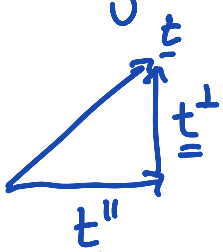
normal stress = $\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel}\,\underline{t}_n = \sigma_n\,\underline{n}$

shear stress : $\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp}\,\underline{t}_n = \tau\,\underline{m}$

$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n}$ or $\boxed{\sigma_n = n_i\,\sigma_{ij}\,n_j}$

$\underline{\tau} = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \underline{\underline{\sigma}}\,\underline{n}$ or $\boxed{\tau = m_i\,\sigma_{ij}\,n_j}$

From geometry:



$$\underline{t} = \underline{t}^{\parallel} + \underline{t}^{\perp}$$

$$|\underline{t}|^2 = |\underline{t}^{\parallel}|^2 + |\underline{t}^{\perp}|^2$$

$$= |\sigma_n \underline{n}|^2 + |\tau \underline{m}|^2 =$$

$$\boxed{|\underline{t}|^2 = \sigma_n^2 + \tau^2}$$

Extremal stress values

→ important for failure

We all know this is related to eigenvalues

But why?

I, Extremal normal stresses

Given any $\underline{\underline{\sigma}}$ at $\underline{x}$ what are the unit normal $\underline{n}$ corresponding to extrema in σ_n?

Constrained optimization problem

find extrema of $\sigma_n = \sigma_n(\underline{n}) = \underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n}$

with constraint that $|\underline{n}| = 1$ $\quad |\underline{n}|^2 = \underline{n} \cdot \underline{n} = 1$

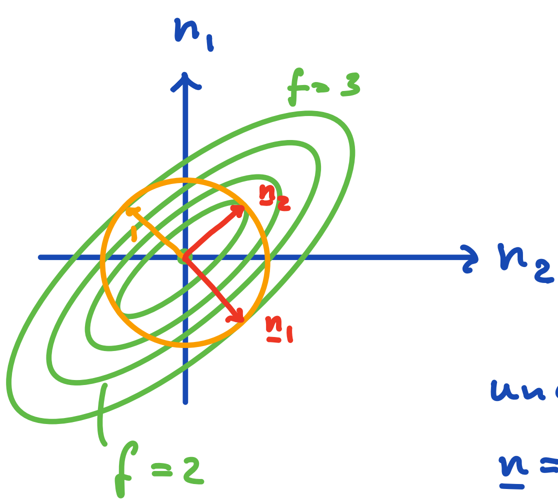
$\underline{n} \cdot \underline{n} - 1 = g(\underline{n})$

⇒ Lagrange multiplier method

$$\boxed{\begin{aligned} \mathcal{L}(\underline{n},\lambda) &= \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n} - \lambda\,(\underline{n}\cdot\underline{n}-1) \\ \mathcal{L}(n_i,\lambda) &= \underbrace{n_i\,\sigma_{ij}\,n_j}_{\text{function to optimize}} - \underbrace{\lambda}_{\text{Lagrange multiplier}}\underbrace{(n_i n_i-1)}_{\text{constraint}} \end{aligned}}$$

Function $f(\underline{n}) = \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n}$ is quadratic

if $\lambda_i > 0$ → level sets are ellipses



$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

$f(\underline{n}) = \underline{0}\cdot\underline{\underline{\sigma}}\,\underline{0} = \underline{0}$

$f(\underline{n}) = \sigma_n(\underline{n}) = \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n}$ -

unconstrained minimum

$\underline{n} = \underline{0}$ $\quad f(\underline{0}) = 0$

Method of Lagrange multipliers

→ extremal values are the stationary points $\mathcal{L}$

$$\mathcal{L}(n_i,\lambda) = n_i\,\sigma_{ij}\,n_j - \underline{\lambda}\,(n_i n_i - 1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_i n_i - 1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij} \left(\frac{\partial n_i}{\partial n_k} n_j + n_i \frac{\partial n_j}{\partial n_k} \right) - \lambda \left(2 n_i \frac{\partial n_i}{\partial n_k} \right) = 0$$

common notation $\frac{\partial n_i}{\partial n_k} = n_{i,k}$

$$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \qquad \frac{\partial n_1}{\partial n_1} = 1 \qquad \frac{\partial n_1}{\partial n_3} = 0 \qquad \frac{\partial n_1}{\partial n_2} = 0$$

$$\Rightarrow \quad n_{i,k} = \delta_{ik} \qquad n_{j,k} = \delta_{jk}$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij} \left(\delta_{ik} n_j - n_i \delta_{jk} \right) - \lambda \left(2 n_i \delta_{ik} \right) = 0$$

$$= \sigma_{kj} n_j - \sigma_{ik} n_i) - \lambda (2 n_k) = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \qquad \sigma_{kj} = \sigma_{jk}$$

$$= \sigma_{jk} n_j - \sigma_{ik} n_i - 2 \lambda n_k$$

rename dummy $j \to i$

$$= 2 \left(\sigma_{ik} n_k - \lambda n_k \right) = 0$$

switch to symbolic notation

$$\underline{\underline{\sigma}} \, \underline{n} - \lambda \underline{n} = 0$$

$$\boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\,\underline{n} = 0}$$

Constrained optimization $\Rightarrow$ eigenvalue problem.

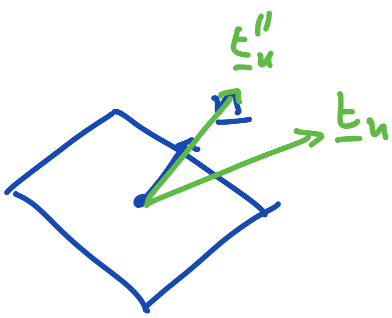
$\Rightarrow$ eigenvectors of $\underline{\underline{\sigma}}$ are the directions of max & min normal stress.

To see that λ's are associated normal stresses you dot eigen problem with $\underline{n}$.

$$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\,\underline{n} = 0$$

$$\underbrace{\underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n}} - \lambda\, \underline{n} \cdot \underline{n} =$$

$$\boxed{\sigma_n = \lambda}$$



$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n}$$

$$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n}$$

λ_i's are principal stresses $\Rightarrow \lambda_i = \sigma_i$

$\sigma_1 > \sigma_2 > \sigma_3$

σ_1 = max normal stress

σ_3 = min normal stress

$\underline{n}_i$'s are the principal directions

What are tractions on principal planes (planes associated $\underline{n}_i$'s)

$$\underline{t}_{\underline{n}_i} = \underline{\underline{\sigma}}\,\underline{n}_i = \left(\sum_{j=1}^{3} \sigma_j\, \underline{n}_j \otimes \underline{n}_j\right) \underline{n}_i = \sum_{j=1}^{3} \sigma_j \underbrace{(\underline{n}_j \cdot \underline{n}_i)}_{\delta_{ij}} \underline{n}_j$$

$$\underline{t}_{n_i} = \sigma_i\, \underline{n}_i$$

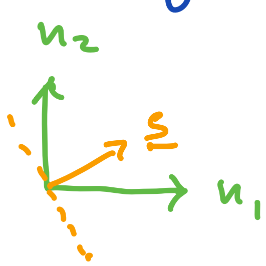
traction is equal to normal stress $\underline{t} \parallel \underline{n}_i$

$\Rightarrow$ no shear stresses on principal planes.

$$\underline{t}^{\perp}_{n_i} = \underline{0}$$

II) Extremal shear stresses

Given the principal dir. $\{\underline{n}_i\}$ at $\underline{x}$ what is the direction/unit vector $\underline{s} = [s_1\ s_2\ s_3]^T$ that gives the max. & min. values of τ ?



$$\boxed{\underline{s} = s_i\, \underline{n}_i} \qquad s_i = \underline{s} \cdot \underline{n}_i$$

traction in dir $\underline{s}$

only 1 index

$$\underline{t}_s = \underline{\underline{\sigma}}\,\underline{s} = \left(\sum_{i=1}^{3} \sigma_i\, \underline{n}_i \otimes \underline{n}_i\right)(s_j n_j)$$

$$= \sum^{3} \sigma_i\, s_i\, (n_i \otimes n_i)\, n_j$$

$$\underbrace{(\underline{n}_i \cdot \underline{n}_j)}_{\delta_{ij}} \underline{n}_i$$

$$= \sum_{i=1}^{3} \sigma_i s_i \underline{n}_i$$

traction on $\underline{s}$

$$\underline{t}_s = \sigma_1 s_1 \underline{n}_1 + \sigma_2 s_2 \underline{n}_2 + \sigma_3 s_3 \underline{n}_3$$

need expression for shear stress τ

$$|\underline{t}_s|^2 = \sigma_n^2 + \tau^2 \Rightarrow \tau^2 = |\underline{t}_s|^2 - \sigma_n^2$$

normal stress: $\sigma_n = \underline{s} \cdot \underline{t}_s = \sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2$

$$\tau^2 = \underline{t}_s \cdot \underline{t}_s - \sigma_n^2 = \sigma_1^2 s_1^2 + \sigma_2^2 s_2^2 + \sigma_3^2 s_3^2 - (\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2)^2$$

shear stress in principal frame:

$$\boxed{\tau^2 = \sum_{i=1}^{3} \sigma_i^2 s_i^2 - \left(\sum_{i=1}^{3} \sigma_i s_i^2\right)^2}$$

$$\tau = \tau(\underbrace{\sigma_1, \sigma_2, \sigma_3}_{\text{given}}, \underbrace{s_1, s_2, s_3}_{\text{unknowns}})$$

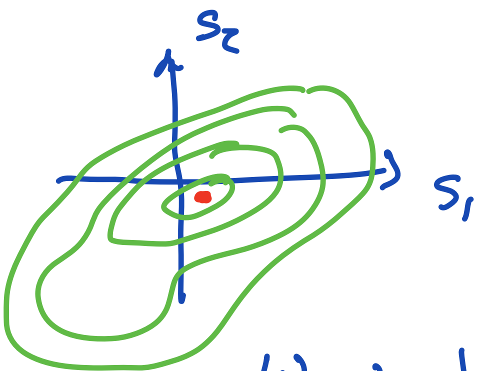
Looking for extrema of τ^2 (τ) under the constraint that $|s| = 1$ $\quad \boxed{s_1^2 + s_2^2 + s_3^2 = 1}$

$$s_3^2 = 1 - s_1^2 - s_2^2$$

Solve constrained optimization problem by direct elimination:

eliminate s_3^2 by substituting $\underline{s_3^2 = 1 - s_1^2 - s_2^2}$

$\Rightarrow \quad \tau^2 = \tau^2(s_1, s_2)$



because we are substuted constraint

$\Rightarrow$ solve for max min directly

We just need to find

$$\frac{\partial \tau^2}{\partial s_1} = \frac{\partial \tau^2}{\partial s_2} = 0$$

$$\frac{\partial \tau^2}{\partial s_1} = 2 s_1 (\sigma_1 - \sigma_3) \left\{ \sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) s_1^2 + (\sigma_2 - \sigma_3) s_2^2 \right] \right\} = 0$$

$$\frac{\partial \tau^2}{\partial s_2} = 2 s_2 (\sigma_2 - \sigma_3) \left\{ \sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) s_1^2 + (\sigma_2 - \sigma_3) s_2^2 \right] \right\} = 0$$

First solution (trivial): $s_1 = s_2 = 0 \Rightarrow s_3 = 1$

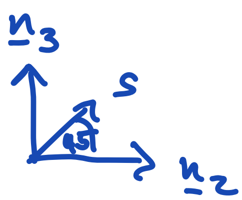
$\Rightarrow \underline{s} = \pm \underline{n}_3$

$\tau^2 = 0$

$\Rightarrow$ minimum in shear stress

on principal planes

Second solution: $s_1 = 0$



$$\Rightarrow \quad s_2 = \pm \frac{1}{\sqrt{2}}$$

$$s_3 = \pm \frac{1}{\sqrt{2}}$$

$$\underline{s} = \pm \frac{1}{\sqrt{2}} (\underline{n}_2 + \underline{n}_3)$$

shear stress: $\tau^2 = \left(\frac{\sigma_2 - \sigma_3}{2} \right)^2 \qquad \tau = \frac{1}{2} (\sigma_2 - \sigma_3)$

We have two solutions:

min: $\tau = 0 \quad$ for $\underline{s} = \pm \underline{n}_3$

max: $\tau = \frac{1}{2} (\sigma_2 - \sigma_3 \quad$ for $\underline{s} = \frac{1}{\sqrt{2}} (\pm \underline{n}_2 \pm \underline{n}_3)$

two additional pairs of solns by eliminating s_1 or s_2 in total we have:

Min. shear stresses:

$\tau = 0 \quad$ on $\quad \underline{s} = \pm \underline{n}_1 , \ \underline{s} = \pm \underline{n}_2 , \ \underline{s} = \pm \underline{n}_3$

Max. shear stresses

$$\tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3) \quad \text{ou} \quad \underline{s}_{23} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3)$$

$$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3) \quad \text{ou} \quad \underline{s}_{13} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_3)$$

$$\tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad \text{ou} \quad \underline{s}_{12} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_2)$$