Lecture 8: Mohr circle & failure

Logistics: - HW2 graded → online

- HW3 due

Last time: - Extremal values of normal & shear stress

→ constrained optimization to find normal to planes

1) Normal stress → eigenvalue problem

$$\boxed{(\underline{\underline{\sigma}} - \sigma_i \underline{\underline{I}})\,\underline{n}_i = 0}$$

→ σ_i principal stresses σ_{ij}

→ $\underline{v}_i$ principal directions

$$\boxed{\underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i \, \underline{n}_i \otimes \underline{n}_i}$$

2) Shear stress

min: $\tau = 0$ on $\underline{s} = \pm \underline{n}_i$

max: $\tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3)$ on $\underline{s} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3)$

$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3)$ on $\underline{s} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_3)$

$\tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2)$ on $\underline{s} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_2)$

Today: Stress representation in Mohr circle

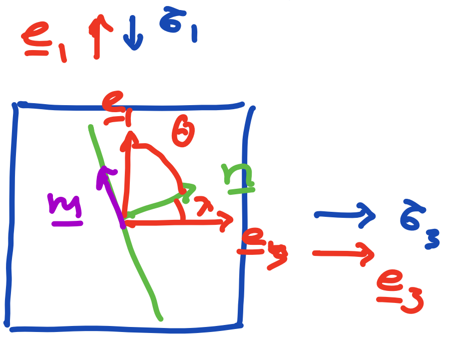
Failure criteria

Geological example

Mohr circle

graphical display of normal & shear stresses

Look at 2D case



θ is angle between $\underline{e}_1$ & $\underline{n}$

λ is angle between $\underline{e}_3$ & $\underline{n}$

$\lambda + \theta = \frac{\pi}{2}$ $(90°)$ $\quad \lambda = \frac{\pi}{2} - \theta$

$$\underline{n} = n_1 \underline{e}_1 + n_3 \underline{e}_3$$

$$n_1 = \underline{n} \cdot \underline{e}_1 = |\underline{n}| \, |\underline{e}_1| \cos\theta = \cos\theta$$

$$n_3 = \underline{n} \cdot \underline{e}_3 = \sin\theta$$

$$\underline{n} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix} \qquad \underline{m} \cdot \underline{n} = \underline{0} \qquad \underline{m} = \begin{pmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

Stress in principal frame

$$\underline{\underline{\sigma}} = \sigma_1 \, \underline{e}_1 \otimes \underline{e}_1 + \sigma_2 \, \underline{e}_2 \otimes \underline{e}_2 + \sigma_3 \, \underline{e}_3 \otimes \underline{e}_3$$

traction: $\underline{t}_n = \underline{\underline{\sigma}}\, \underline{n} = \sigma_1 \underbrace{\cos\theta}_{\underline{n}\cdot\underline{e}_1} \underline{e}_1 + \sigma_3 \sin\theta \, \underline{e}_3$

normal stress: $\sigma_n = \underline{n} \cdot \underline{t}_n = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$

use $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ $\quad \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\Rightarrow \boxed{\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cos(2\theta)}$$

shear stress: $\tau = \underline{m} \cdot \underline{t}_n = (\sigma_1 - \sigma_3) \sin\theta \cos\theta$

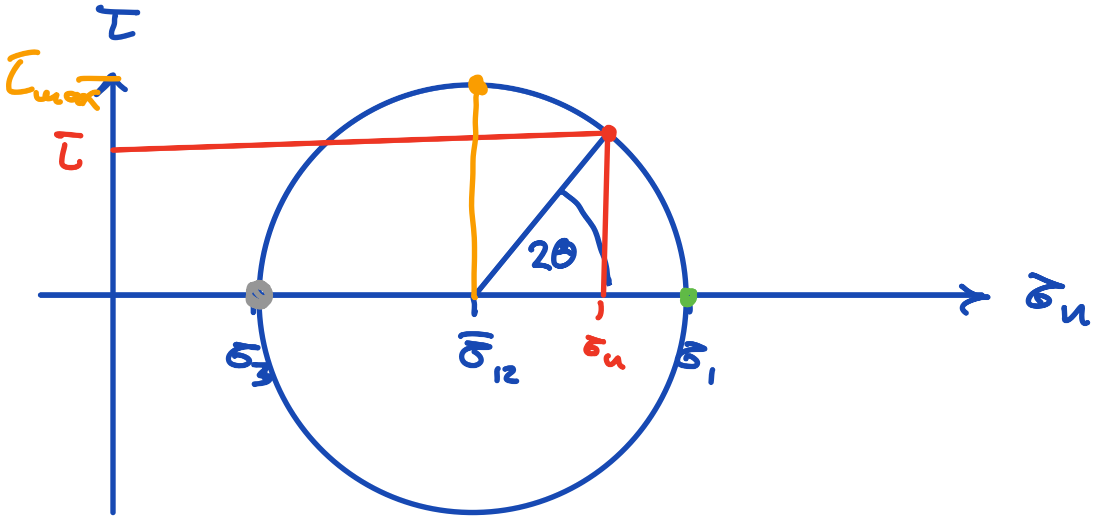
use $2 \sin\theta \cos\theta = \sin 2\theta$

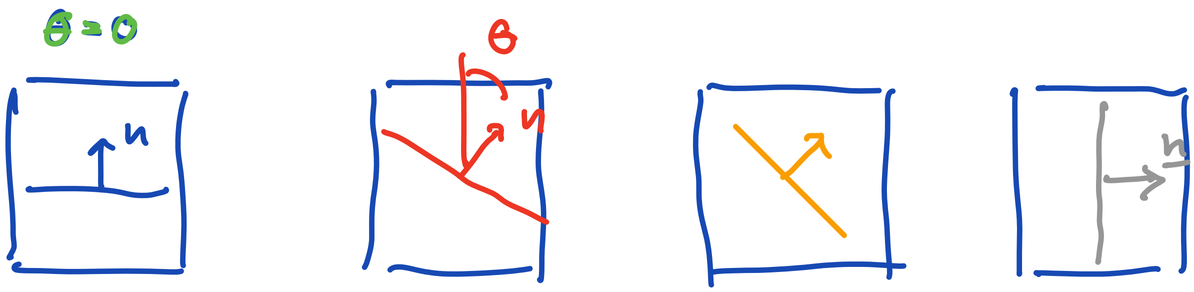
$$\Rightarrow \boxed{\tau = \frac{\sigma_1 - \sigma_3}{2} \sin 2\theta}$$

Eqns for circle in $\sigma_n \tau$ with center

$\left(\dfrac{\sigma_1 + \sigma_3}{2}, 0\right)$ with radius $\dfrac{\sigma_1 - \sigma_3}{2}$

Note: $\bar{\sigma}_{13} = \dfrac{\sigma_1 + \sigma_3}{2}$ $\qquad \tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2}$





$\sigma_n = \sigma_1$	$\sigma_3 < \sigma_n < \sigma_1$	$\sigma_n = \bar{\sigma}_{13}$	$\sigma_n = \sigma_3$
$\tau = 0$	$\tau > 0$	$\tau = \tau_{max}$	$\bar{\tau} = 0$

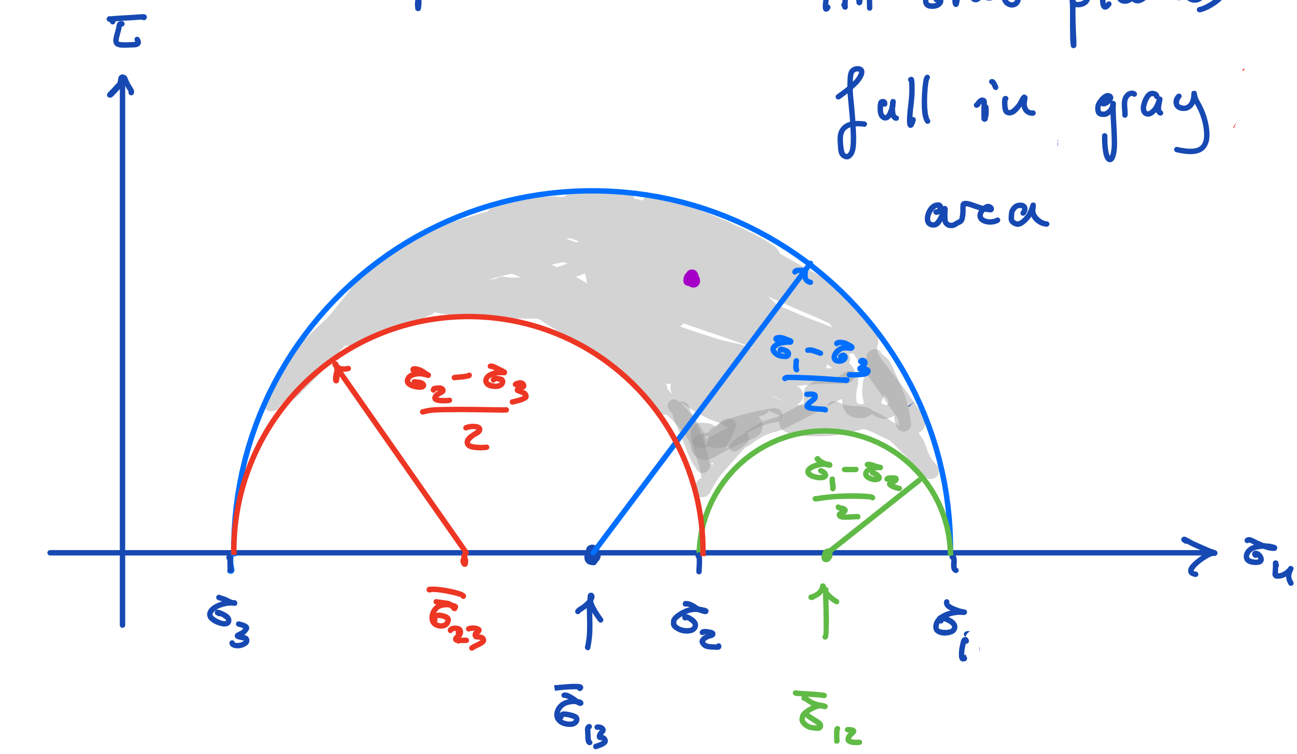
$\Rightarrow$ this for plane that is parallel to $\underline{e}_2$

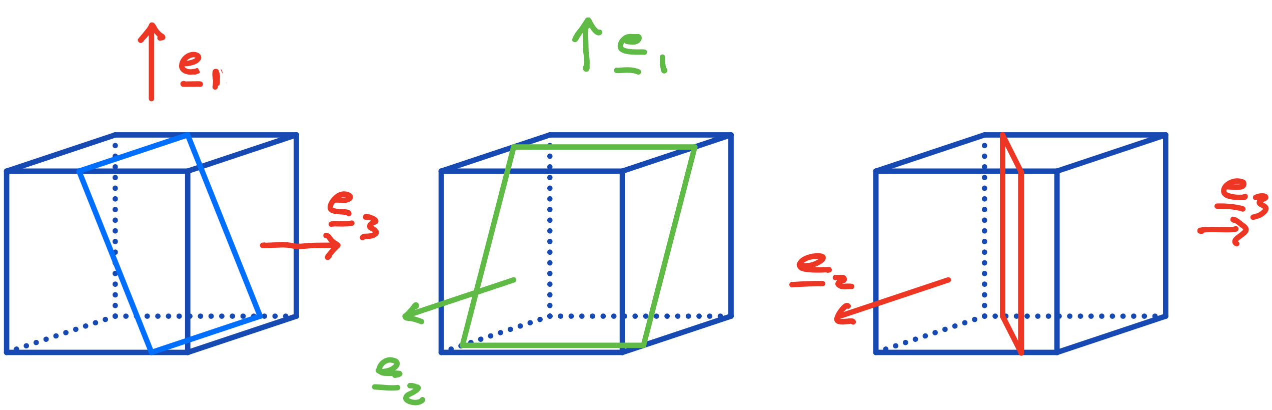
Mohr circle in 3D

Repeat these arguments for planes parallel to $\underline{e}_1$ or $\underline{e}_3$.

$\Rightarrow$ Two additional circles for the σ_n & τ on those planes.

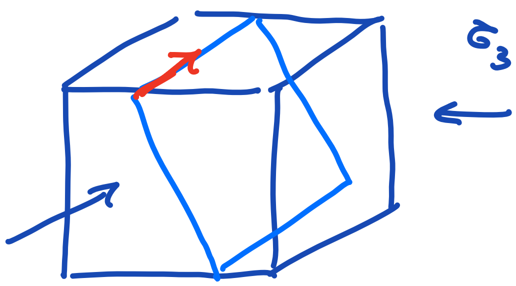
All other planes fall in gray area



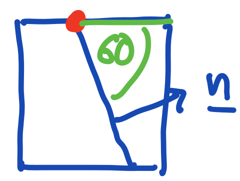


Example: Normal & Shear stress on a fault
(N. Espinoza)

$\sigma_1 = \sigma_v = 23$ MPa



$\sigma_3 = \sigma_h = 13.8$ MPa



$\sigma_2 = \sigma_H = 20$ MPa

Fault orientation:

strike: azimuth $0°$

dip: $60°$

Note: Neglect pore pressure

mean stress: $\bar{\sigma}_{13} = \dfrac{23 + 13.8}{2} = 18.4$ MPa

$\tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2} = 4.6$ MPa

normal stress: $\sigma_n = \bar{\sigma}_{13} + \tau_{max} \cos(2 \cdot 60°) = 16.1$ MPa

shear stress: $\tau = \tau_{max} \sin(2 \cdot 60°) = 4.0$ MPa

More general way → change of basis tensor

⇒ switch to notes

Failure criteria for shear fracture

most common type of failure

⇒ empirical criteria

I Tresca criterion

Fracture occurs if $\tau_{max} = \tau_{13} = \frac{\sigma_1 - \sigma_3}{2}$

reaches the shear yield strength σ_Y

$$\boxed{|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \sigma_Y}$$

Note: Not affected by mean stress

⇒ predicts conjugate shear planes

at $45°$ to principal dir.

II Coulomb criterion

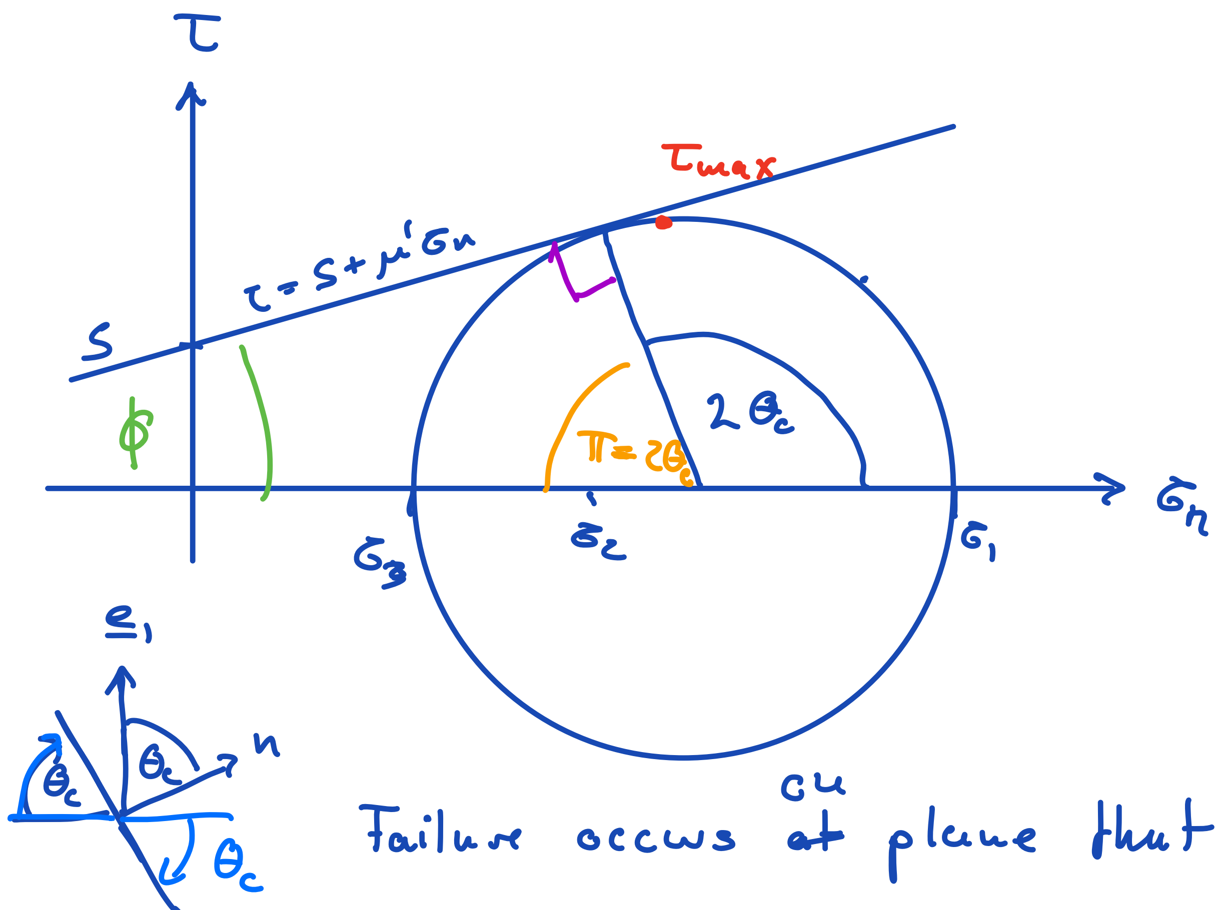
Fracture depends on both mag. shear stress and the normal stress

$$\boxed{|\tau| = S + \mu' \sigma_n}$$

S = cohesive strength $\sim$ 10 – 100 MPa

$\mu' = \tan\phi$ internal friction ~~eq~~ ~ 0.6

$\phi = 30°$ angle of internal friction



Failure occurs ~~at~~ on plane that does not experience τ_{max}

$$\phi - \frac{\pi}{2} + (\pi - 2\theta_c) = \pi$$

$$\Rightarrow \theta_c = \underset{45^\circ}{\frac{\pi}{4}} + \underset{15^\circ}{\frac{\phi}{2}} = 60^\circ$$