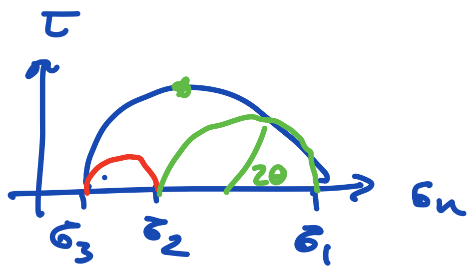
Lecture 9: Tensor calculus

Logistics: - HW3 ✓ is graded

- HW4 will be posted

- next week class is online (travel)

Last time: - Mohr circle



Graphical representation of σ_n & τ

on planes parallel to principal directions

- Shear failure $\qquad \mu' = \tan\phi$

Mohr-Coulomb: $|\tau| = S + \mu' \sigma_n$

Today: Tensor calculus

Div, Grad, Curl and all that!

$\Rightarrow$ Differentiation of tensor fields

Differentiation of tensor fields

A field is a function of space.

scalar fields:	$\phi(\underline{x})$	temp, density
vector fields:	$\underline{v}(\underline{x})$	force, velocity
tensor fields:	$\underline{\underline{S}}(\underline{x})$	stress, conductivity

$\Rightarrow$ review & extension of multivariable calculus

Gradient

Gradient of a scalar field

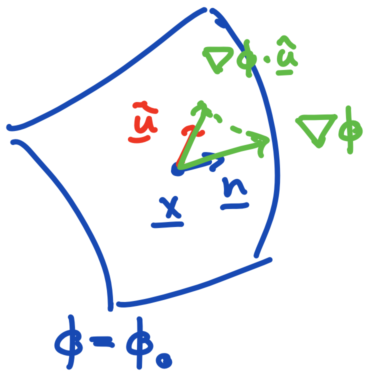
Scalar field $\phi(\underline{x})$ is differentiable at $\underline{x}$ if there exists a vector field $\nabla\phi(\underline{x})$ s.t.

$$\phi(\underline{x}+\underline{h}) = \phi(\underline{x}) + \nabla\phi(\underline{x})\cdot\underline{h} + \text{h.o.t.}$$

by Taylor expansion. $\underline{h} = \epsilon\,\hat{\underline{u}}$ $\quad |\hat{\underline{u}}|=1$ $\quad \epsilon \ll 1$

ϵ ε

$$\boxed{\nabla\phi(\underline{x})\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\,\phi(\underline{x}+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0}}$$

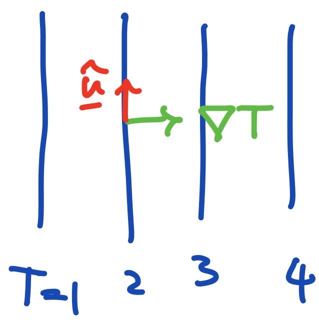


Vector $\nabla\phi$ is called the gradient of ϕ. $\nabla\phi \parallel \hat{n}$ of level sets of ϕ points in direction of increasing ϕ.

Fourier's law: $q = -k\,\nabla T$

Directional derivative:

$$\boxed{D_{\hat{\underline{u}}}\phi(\underline{x}) = \frac{d}{d\epsilon}\phi(\underline{x} + \epsilon\hat{\underline{u}})\Big|_{\epsilon=0} = \nabla\phi(\underline{x})\cdot\hat{\underline{u}}}$$



$D_{\hat{\underline{u}}}T(\underline{x}) = 0$

Representation of gradient in $\{\underline{e}_i\}$

$$\phi(\underbrace{\bar{\underline{x}} + \epsilon\hat{\underline{u}}}_{\underline{x}}) = \phi(\underbrace{\bar{x}_1 + \epsilon\hat{u}_1}_{x_1}, \underbrace{\bar{x}_2 + \epsilon\hat{u}_2}_{x_2}, \underbrace{\bar{x}_3 + \epsilon\hat{u}_3}_{x_3})$$

$$\nabla\phi\cdot\hat{u} = \frac{d}{d\epsilon}\phi(\bar{x}_1 + \epsilon\hat{u}_1, \bar{x}_2 + \epsilon\hat{u}_2, \bar{x}_3 + \epsilon\hat{u}_3)\Big|_{\epsilon=0}$$

$$= \frac{\partial\phi}{\partial x_1}\frac{dx_1}{d\epsilon} + \frac{\partial\phi}{\partial x_2}\frac{dx_2}{d\epsilon} + \frac{\partial\phi}{\partial x_3}\frac{dx_3}{d\epsilon}$$

$$\frac{d x_1}{d\epsilon} = \frac{d}{d\epsilon}(x_1 + \epsilon \hat{u}_1) = \hat{u}_1$$

$$= \frac{\partial \phi}{\partial x_1}\hat{u}_1 + \frac{\partial \phi}{\partial x_2}\hat{u}_2 + \frac{\partial \phi}{\partial x_3}\hat{u}_3$$

$$= \underbrace{\frac{\partial \phi}{\partial x_i}}_{\phi_{,i}}\hat{u}_i = \phi_{,i}\,\hat{u}_i = (\phi_{,i}\,\underline{e}_i)\cdot(\hat{u}_j\,\underline{e}_j)$$

Gradient in components : $\boxed{\nabla\phi = \phi_{,i}\,\underline{e}_i = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix}}$

Gradient of a vector field

A vector field is differentiable at $\underline{x}$ if there exists a tensor field $\nabla\underline{v}(\underline{x})$ s.t.

$$\underline{v}(\underline{x} + \underline{h}) = \underline{v}(\underline{x}) + \nabla\underline{v}(\underline{x})\underline{h} + \text{h.o.t.}$$

by Taylor expansion $\quad \underline{h} = \epsilon\,\hat{\underline{u}}$

$$\boxed{\nabla\underline{v}\,\hat{\underline{u}} = \frac{d}{d\epsilon}\,\underline{v}(\underline{x} + \epsilon\,\hat{\underline{u}})\Big|_{\epsilon=0}}$$

In frame $\{\underline{e}_i\}$ the components of $\underline{v}$ are

$$v_i = v_i(x_1, x_2, x_3)$$

$$v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = v_i(\underbrace{\bar{x}_1 + \epsilon \hat{u}_1}_{x_1}, \underbrace{\bar{x}_2 + \epsilon \hat{u}_2}_{x_2}, \underbrace{\bar{x}_3 + \epsilon \hat{u}_3}_{x_3})$$

by chain rule

$$\frac{d}{d\epsilon} v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = \frac{\partial v_i}{\partial x_1}\hat{u}_1 + \frac{\partial v_i}{\partial x_2}\hat{u}_2 + \frac{\partial v_i}{\partial x_3}\hat{u}_3$$

$$= \frac{\partial v_i}{\partial x_j}\hat{u}_j = v_{i,j}\hat{u}_j$$

Derivative of full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \hat{\underline{u}} = \frac{d}{d\epsilon} \underline{v}(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\Big|_{\epsilon=0} = \frac{d}{d\epsilon} v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\, \underline{e}_i \Big|_{\epsilon=0}$$

$$= v_{i,j}\hat{u}_j e_i$$

$$\boxed{\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_3^T \end{bmatrix}$$

Divergence of a vector field

To any $\underline{v}(x)$ we associate a scalar field

$\nabla \cdot \underline{v}$ called the divergence of $\underline{v}$

$$\boxed{\nabla \cdot \underline{v} = tr(\nabla \underline{v})}$$

In frame $\{\underline{e}_i\}$ $\underline{v}(\underline{x}) = v_i(x)\, \underline{e}_i$

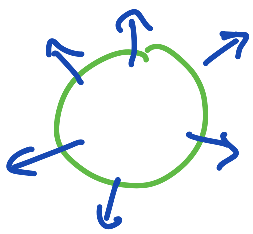
$$\nabla \cdot \underline{v} = tr(\nabla \underline{v}) = tr(v_{i,j}\, \underline{e}_i \otimes \underline{e}_j) = v_{i,i}$$

$$= v_{1,1} + v_{2,2} + v_{3,3}$$

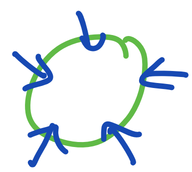
If $\nabla \cdot \underline{v} = 0$ a field is solenoidal or divergence free

If $\underline{v}$ is displacement or velocity

$\nabla \cdot \underline{v}$ is related to volume change



$\nabla \cdot v > 0$ expansion



$\nabla \cdot v < 0$ compression

$\nabla \cdot \underline{v} = 0$ incompressible material

Divergence of a tensor field

To any $\underline{\underline{S}}(x)$ we associate a vector field $\nabla \cdot \underline{\underline{S}}$ called divergence of $\underline{\underline{S}}$

s.t. $\boxed{(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})}$

uses the definition of the vector divergence !

In frame $\{\underline{e}_i\}$ $\quad \underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j \quad \underline{a} = a_k \underline{e}_k$

$\underline{q} = \underline{\underline{S}}^T \underline{a} \qquad q_j = S_{ij}\, a_i$

s-bstitute

$$(\nabla\cdot \underline{\underline{S}})\cdot \underline{a} = \nabla\cdot(\underline{\underline{S}}^T\underline{a}) = \nabla\cdot q = \mathrm{tr}(\nabla \underline{q}) = q_{j,j}$$

$$= (S_{ij}\, a_i)_{,j} = S_{ij,j}\, a_i + S_{ij}\, \cancel{a_{i,j}} \to 0$$

$$= S_{ij,j}\, a_i = \underbrace{(S_{ij,j}\, \underline{e}_i)}_{\nabla\cdot\underline{\underline{S}}}\cdot(a_k \underline{e}_k)$$

by arbitrariness of $\underline{a}$

$$\boxed{\nabla\cdot\underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence Product Rules

$\phi(\underline{x}) \in \mathbb{R}$ $\quad \underline{v}(\underline{x}) \in \mathcal{V}$ $\quad \underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$

$$\nabla\cdot(\phi\underline{v}) = \underline{v}\cdot\nabla\phi + \phi\,\nabla\cdot\underline{v}$$

$$\nabla\cdot(\phi\underline{\underline{S}}) = \underline{\underline{S}}\,\nabla\phi + \phi\,\nabla\cdot\underline{\underline{S}}$$

$$\nabla\cdot(\underline{\underline{S}}^T\underline{v}) = (\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}} : \nabla\underline{v}$$

$$\nabla(\phi\underline{v}) = \underline{v}\otimes\nabla\phi + \phi\,\nabla\underline{v}$$

Example : $\nabla\cdot(\underline{\underline{S}}^T\underline{v})$ $\qquad \underline{\underline{S}} = \underline{\underline{S}}(\underline{x}) \qquad \underline{v} = \underline{v}(\underline{x})$

$$q(\underline{x}) = \underline{\underline{S}}^T\underline{v} \qquad q_j = S_{ij}\, v_i$$

$$\nabla \cdot \underset{\sim}{q} = \mathrm{tr}(\underset{\sim}{q}) = q_{j,j} = (S_{ij} v_i)_{,j}$$

$$= S_{ij,j} v_i + S_{ij} v_{i,j}$$

$$= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v}$$

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij} B_{ij}$$

Curl of a vector field

To any $\underline{v}(x)$ we associate another vector field $\nabla \times \underline{v}$ called the curl of $\underline{v}$

s.t. $\boxed{(\nabla \times \underline{v}) \times \underline{a} = (\nabla \underline{v} - \nabla \underline{v}^T)\, \underline{a}}$

$$\underline{\underline{T}} = \nabla \underline{v} - \nabla \underline{v}^T = 2\, \mathrm{skew}(\nabla \underline{v}) \qquad \begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{23} & 0 & T_{23} \\ -T_{13} & -T_{12} & 0 \end{bmatrix}$$

$\underline{\omega} = \nabla \times \underline{v}$ is axial vector of $\underline{\underline{T}}$

$$\omega_j = \tfrac{1}{2} \epsilon_{ijk} T_{ik} = \tfrac{1}{2} \epsilon_{ijk} (v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}) \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad \text{flip } i \leftrightarrow k \text{ second term}$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{ijk} v_{i,k})$$

$$\omega_j = \epsilon_{ijk} v_{i,k}$$

$$\Rightarrow \quad \boxed{\underline{w} = \nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j} = -\epsilon_{ijk} v_{i,j} \underline{e}_k$$

$$\underline{a} \times \underline{b} = \epsilon_{ijk} a_i b_j \underline{e}_k$$

Explicit: $\nabla \times \underline{v} = (v_{3,2} - v_{2,3}) \underline{e}_1 + (v_{1,3} - v_{3,1}) \underline{e}_2$
$$+ (v_{2,1} - v_{1,2}) \underline{e}_3$$

Physical interpretation:

If $\underline{v}$ is a velocity field $\nabla \times \underline{v}$ measures the angular velocity

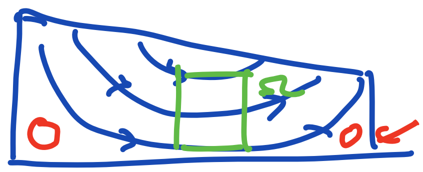
If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}(\underline{x})$ is irrotational/conservative

Important div-curl relationships:

$$\boxed{\nabla \times \nabla \phi = 0} \quad \text{and} \quad \boxed{\nabla \cdot (\nabla \times \underline{v}) = 0}$$

Example: Darcy's law $q = -K \nabla h$

$$\nabla \times q = \nabla \times (-K \nabla h) = -K \nabla \times \nabla h = 0$$



~~tan~~ cannot be $\Rightarrow$ numerical errors

$\nabla \cdot (\nabla \times q) = 0$ trivial

$$\int_{\Omega} \nabla \cdot \underline{q} \, dV = \oint \underline{q} \cdot n \, dS = 0$$