Infinitesimal Strain Tensor

Conider deformation $\varphi : \mathcal{B} \to \mathcal{B}'$ with the associated displacement field $\underline{u} = \varphi(\underline{X}) - \underline{X}$ and displacement gradient $\nabla \underline{u}$. Then another measure of strain is provided by

$$\boxed{\underline{\underline{\varepsilon}} = \operatorname{sym}(\nabla \underline{u}) = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)}$$

$\underline{\underline{\varepsilon}} : \mathcal{B} \to \mathcal{V}^2$ is the infinitesimal strain tensor field associated with φ. By definition $\underline{\underline{\varepsilon}}$ is symmetric.

To relate ∇u to $\underline{\underline{F}}$ and $\underline{\underline{C}}$ consider

$$\nabla \underline{u} = \nabla(\varphi(\underline{X}) - \underline{X}) = \nabla\varphi(\underline{X}) - \underline{\underline{I}} = \underline{\underline{F}} - \underline{\underline{I}}$$

hence

$$\boxed{\underline{\underline{\varepsilon}} = \operatorname{sym}(\underline{\underline{F}} - \underline{\underline{I}}) = \frac{1}{2}\left(\underline{\underline{F}} + \underline{\underline{F}}^T\right) - \underline{\underline{I}}}$$

Given that $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ and $\underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}}$ we have

$$\underline{\underline{C}} = (\nabla \underline{u} + \underline{\underline{I}})^T (\nabla \underline{u} + \underline{\underline{I}}) = (\nabla \underline{u}^T + \underline{\underline{I}})(\nabla \underline{u} + \underline{\underline{I}})$$

$$= \nabla \underline{u}^T \nabla \underline{u} + \underbrace{\nabla \underline{u} + \nabla \underline{u}^T}_{2\underline{\underline{\varepsilon}}} + \underline{\underline{I}}$$

$$\Rightarrow \quad \boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) - \frac{1}{2} \nabla \underline{u}^T \nabla \underline{u} = \underline{\underline{E}} - \frac{1}{2} \nabla \underline{u}^T \nabla \underline{u}}$$

The tensor $\underline{\underline{\varepsilon}}$ is useful in the case of small deformations.

We say φ is small if $|\nabla \underline{u}| = \mathcal{O}(\varepsilon)$ for all $\underline{X} \in B$ where $0 < \varepsilon \ll 1$. In this case,

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) + \mathcal{O}(\varepsilon^2) \; .$$

If terms of $\mathcal{O}(\varepsilon^2)$ are neglected $\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}$

For small deformations $\underline{\underline{\varepsilon}}$ contains the same information as $\underline{\underline{C}}$, but $\underline{\underline{\varepsilon}}$ is linear function of $\underline{u}$ while $\underline{\underline{C}}$ is a non-linear function of $\underline{u}$. The tensor $\underline{\underline{\varepsilon}}$ arises in linearized models of stress in elastic solids.

Interpretation of the components of $\underline{\underline{\varepsilon}}$

Let ε_{ij} be the components of $\underline{\underline{\varepsilon}}$ in frame $\{\underline{e}_i\}$ and assume a small deformation so that we neglect terms of $\mathcal{O}(\varepsilon^2)$. Then for any $\underline{X} \in B$ we have

$\boxed{\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1}$ and $\boxed{\varepsilon_{ij} \approx \frac{1}{2} \sin \gamma(\underline{e}_i, \underline{e}_j)}$ $i \neq j$, no sum

where $\lambda(\underline{e}_i)$ is the strech in dir. $\underline{e}_i$ and $\gamma(\underline{e}_i, \underline{e}_j)$ is the shear between directions $\underline{e}_i$ and $\underline{e}_j$.

For the diagonal components consider

$$C_{ii} = 1 + 2\varepsilon_{ii} + \mathcal{O}(\varepsilon^2) \qquad \text{no sum}$$

since $\varepsilon_{ii} = \mathcal{O}(\varepsilon)$ and $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$ (Taylor Series)

$$\sqrt{C_{ii}} = \sqrt{1 + \underbrace{2\varepsilon_{ii}}_{x}} = 1 + \varepsilon_{ii} + \mathcal{O}(\varepsilon^2)$$

neglecting terms of order $\mathcal{O}(\varepsilon^2$

$$\varepsilon_{ii} \approx \sqrt{C_{ii}} - 1 = \lambda(\underline{e}_i) - 1 \checkmark \qquad \text{no sum}$$

The strech $\lambda(\underline{e}_i)$ is limit of $\frac{|\underline{y}-\underline{x}|}{|\underline{Y}-\underline{X}|}$ so that

$$\lambda(\underline{e}_i) - 1 = \frac{|\underline{y}-\underline{x}| - |\underline{Y}-\underline{X}|}{|\underline{Y}-\underline{X}|} \qquad \text{relative change in length}$$

For the off-diagonal components we have

$$\sin\gamma(\underline{e}_i, \underline{e}_j) = \frac{C_{ij}}{\sqrt{C_{ii}}\sqrt{C_{jj}}} \quad \text{(no sum)}$$

from def. of $\underline{\underline{\varepsilon}}$ we have

$$C_{ij} = 2\varepsilon_{ij} + \mathcal{O}(\varepsilon^2) \quad i \neq j$$

$$C_{ii} = 1 + \mathcal{O}(\varepsilon) \quad \text{(no sum)} \quad \text{since } \varepsilon_{ii} = \mathcal{O}(\varepsilon)$$

risky!

so that $\sqrt{C_{ii}}\sqrt{C_{jj}} = (1+\mathcal{O}(\varepsilon))(1+\mathcal{O}(\varepsilon)) = 1 + \mathcal{O}(\varepsilon^2)$ (no sum)

substituting we obtain $\sin\gamma(\underline{e}_i, \underline{e}_j) = 2\underline{\underline{\varepsilon}}_{ij} + \mathcal{O}(\varepsilon^2)$.

Hence neglecting terms of $\mathcal{O}(\varepsilon^2)$ we have

$$\varepsilon_{ij} = \frac{1}{2}\sin\gamma(\underline{e}_i, \underline{e}_j) \checkmark$$

When the shear angle is small, then

$$\varepsilon_{ij} \approx \frac{1}{2}\sin\gamma(\underline{e}_i, \underline{e}_j) \approx \frac{1}{2}\gamma(\underline{e}_i, \underline{e}_j) \quad i \neq j$$

$\Rightarrow$ ε_{ij} is half the shear angle between coordinate directions.

Green-Lagrange strain tensor

The tensor $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ is the non-linear extension of $\underline{\underline{\varepsilon}}$. Here there is no notion that $|\nabla \underline{u}| \ll 1$. Because $\underline{\underline{E}}$ limits naturally to $\underline{\underline{\varepsilon}}$ it is a popular choice to extend constitutive laws from small to finite deformations.

Linearization of Kinematic Quantities

Given a deformation $\underline{x} = \underline{\varphi}(\underline{X})$ and the displacement field $\underline{u} = \underline{x} - \underline{X}$ we have the displacement gradient, $\underline{\underline{H}} = \nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}}$

We are interested in the linearization of the tensor fields: $\underline{\underline{U}}, \underline{\underline{V}}, \underline{\underline{R}}, \underline{\underline{C}}, \underline{\underline{E}}$ in the limit when $\underline{\underline{H}}$ is small.

Norm: $|\underline{\underline{H}}| = \sqrt{\underline{\underline{H}} : \underline{\underline{H}}} = (H_{11}^2 + H_{12}^2 + \ldots + H_{33}^2)^{1/2} = \epsilon$

if $|\underline{\underline{H}}| \to 0$ then each component $H_{ij} \to 0$

Let $\underline{\underline{Z}}(\underline{\underline{H}})$ be a tensor-valued tensor function of $\underline{\underline{H}}$. We say $\underline{\underline{Z}}(\underline{\underline{H}}) = \mathcal{O}(|\underline{\underline{H}}|^n)$ as $|\underline{\underline{H}}| \to 0$ if there exists a number $\alpha > 0$ such that

$$|\underline{\underline{Z}}(\underline{\underline{H}})| < \alpha |\underline{\underline{H}}|^n \text{ as } |\underline{\underline{H}}| \to 0$$

Using Taylor expansion in principal basis it can be shown that for any sym. $\underline{\underline{A}}$ and $m \in \mathbb{R}$ we have

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\underline{\underline{A}} + \mathcal{O}(|A|^2) \quad \text{as } |\underline{\underline{A}}| \to 0$$

Using this we can show that as $|\underline{\underline{H}}| = \epsilon \to 0$

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + \mathcal{O}(\epsilon^2)$$

$$\underline{\underline{V}}^2 = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + \mathcal{O}(\epsilon^2)$$

$$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + \mathcal{O}(\epsilon^2)$$

$$\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + \mathcal{O}(\epsilon^2)$$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) + \mathcal{O}(\epsilon^2)$$

where we identify the two tensors

$$\boxed{\begin{aligned} \underline{\underline{\varepsilon}} &= \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) = \mathrm{sym}(\nabla \underline{u}) \\ \underline{\underline{\omega}} &= \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) = \mathrm{skew}(\nabla \underline{u}) \end{aligned}}$$

infinit. strain ten.

infinit. rotation tens.

Example: $\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = (\underline{\underline{H}} + \underline{\underline{I}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}})^{-1}$

$$= (\underline{\underline{H}} + \underline{\underline{I}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}} + \cancelto{0}{\mathcal{O}(\varepsilon^2)})$$

$$= \underline{\underline{H}} - \cancelto{\varepsilon^2}{\underline{\underline{H}}\,\underline{\underline{\varepsilon}}} + \underline{\underline{I}} - \underline{\underline{\varepsilon}}$$

$$= \underline{\underline{I}} + \underline{\underline{H}} - \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T)$$

$$= \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) \quad \checkmark$$

Decomposition into strech & rotation

$$\underline{\underline{F}} = \underline{\underline{H}} + \underline{\underline{I}} = \underline{\underline{I}} + \mathrm{sym}(\underline{\underline{H}}) + \mathrm{skew}(\underline{\underline{H}})$$

$$= \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$$

for infinitesimal deformations strech and rotation are additive: $\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$

For finite deformations strech and rotation are multiplicative $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

$$\underline{\underline{F}} = (\underline{\underline{I}} + \underline{\underline{\omega}} + \cancelto{0}{\mathcal{O}(\varepsilon^2)})(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \cancelto{0}{\mathcal{O}(\varepsilon^2)})$$

$$\underline{\underline{F}} \approx \underline{\underline{I}} + \underline{\underline{\omega}} + \underline{\underline{\varepsilon}} + \cancelto{\varepsilon^4 \to 0}{\underline{\underline{\omega}}\,\underline{\underline{\varepsilon}}}$$