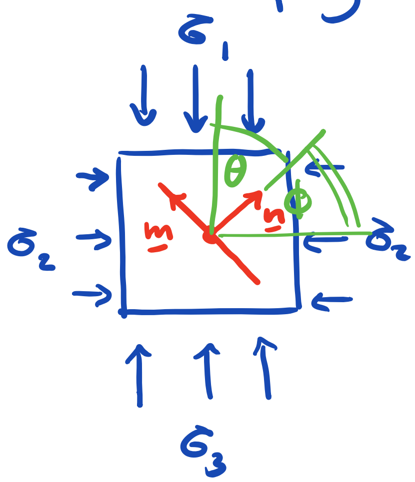
Mohr circle

see Newman for 3D Mohr circle

Mohr circle is a graphical way to display the normal and shear stress on all planes.

For simplicity we look at 2D case, which is already very useful in geology.

Consider physical plane containing σ_1 and σ_3



θ angle between $\underline{n}$ and $\underline{e}_1$

λ angle between $\underline{n}$ and $\underline{e}_3$

$\lambda + \theta = \frac{\pi}{2} \quad \Rightarrow \quad \lambda = \frac{\pi}{2} - \theta$

$$\underline{n} = n_1 \underline{e}_1 + n_2 \underline{e}_3$$

$$n_1 = \underline{n} \cdot \underline{e}_1 = |\underline{n}|\,|\underline{e}_1| \cos\theta = \cos\theta$$

$$n_2 = \underline{n} \cdot \underline{e}_2 = \sin\theta$$

$$\Rightarrow \quad \underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \Rightarrow \quad \underline{m} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Stress in principal frame $\{\underline{e}_i\}$

$$\underline{\underline{\sigma}} = \sigma_1\, \underline{e}_1 \otimes \underline{e}_1 + \sigma_2\, \underline{e}_2 \otimes \underline{e}_2 + \sigma_3\, \underline{e}_3 \otimes \underline{e}_3$$

traction: $\underline{t}_n = \underline{\underline{\sigma}}\, n = \sigma_1 \cos\theta\, \underline{e}_1 + \sigma_3 \sin\theta\, \underline{e}_3$

normal stress: $\sigma = \underline{n} \cdot \underline{t}_n = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$

use: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\Rightarrow \boxed{\sigma = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cos 2\theta}$$

shear stress: $\tau = \underline{m} \cdot \underline{t}_n = (\sigma_1 - \sigma_3) \sin\theta \cos\theta$

use $2 \sin\theta \cos\theta = \sin 2\theta$

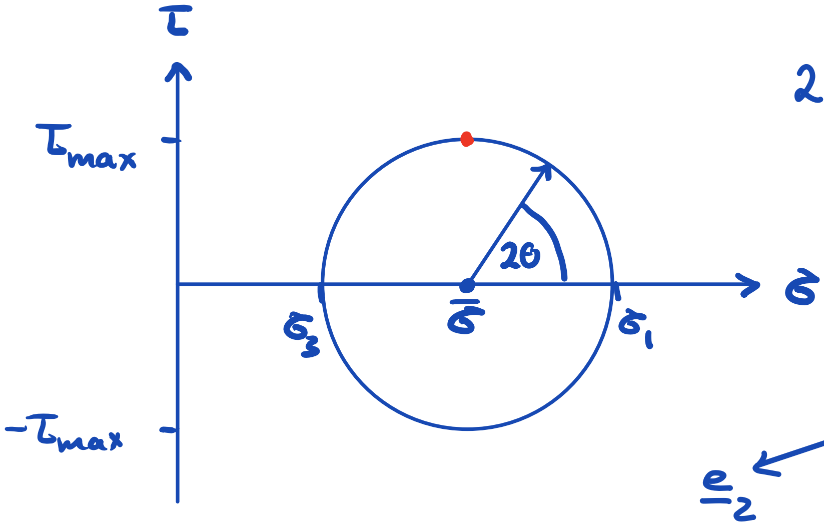
$$\Rightarrow \boxed{\tau = \frac{\sigma_1 - \sigma_3}{2} \sin 2\theta}$$

Together these are equations for circle in $\tau\sigma$-space with radius $R = \frac{\sigma_1 - \sigma_3}{2}$ and center $\left(\frac{\sigma_1+\sigma_3}{2}, 0\right)$

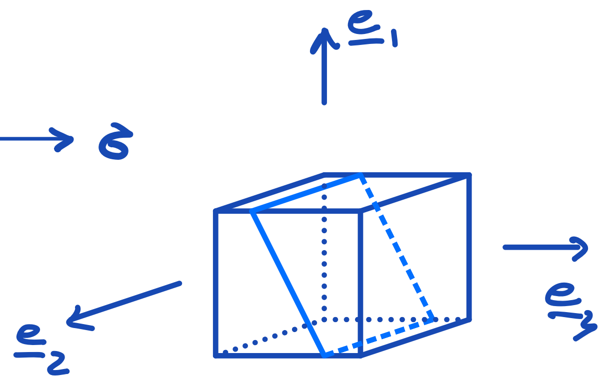
Note: max shear stress: $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = R$

mean stress: $\bar{\sigma} = \frac{\sigma_1 + \sigma_3}{2}$

For Mohr circle construction compressive stresses are assumed to be positive!



2D Mohr circle



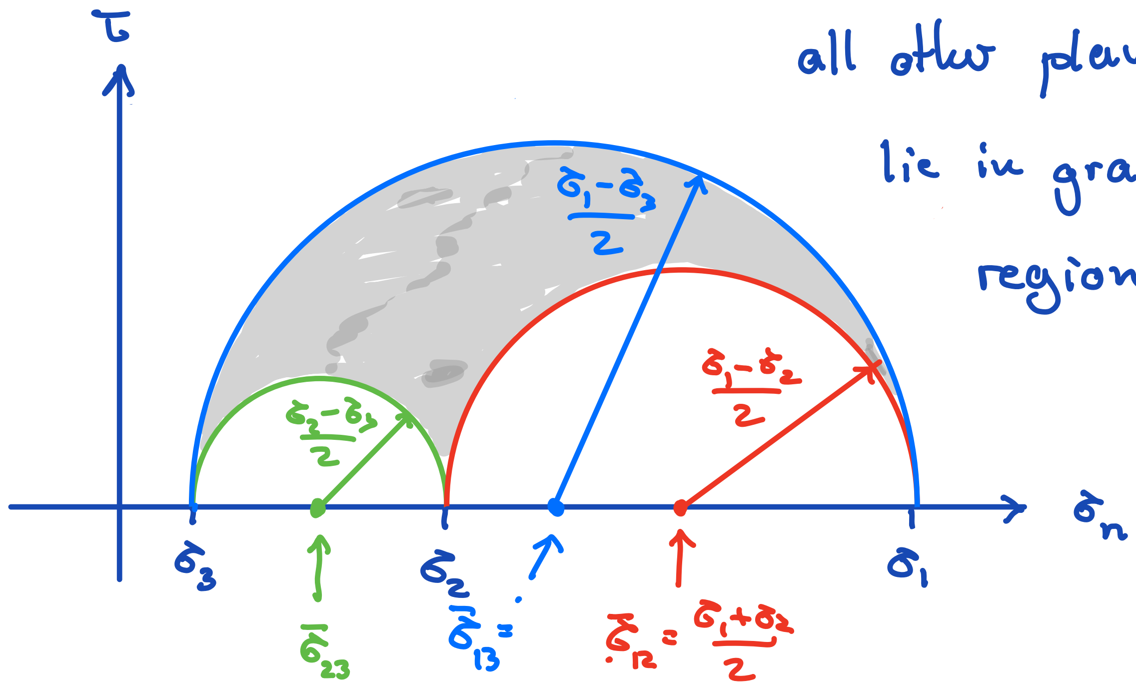
This is another way of showing that the max. shear stress is at 45° to $\underline{n}_1$ and $\underline{n}_3$.

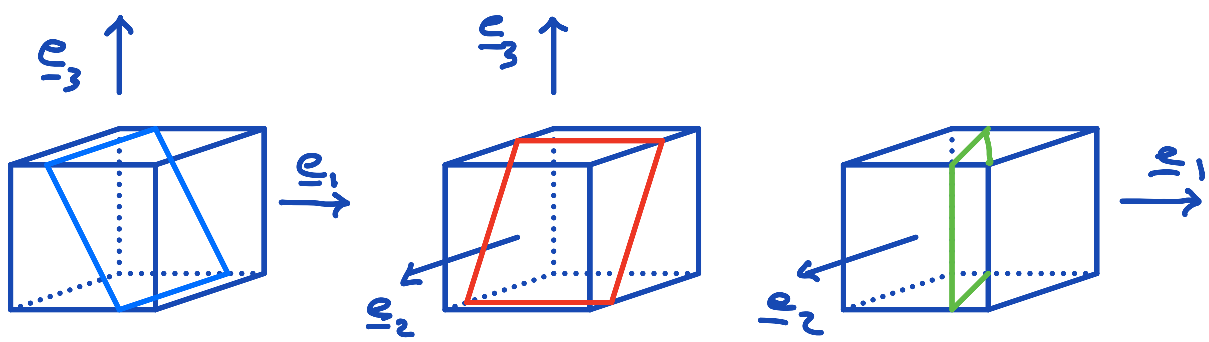
$\Rightarrow$ plane parallel to $\underline{e}_2$

Mohr circles in 3D

Repeat the arguments above for planes parallel to $\underline{e}_1$ and $\underline{e}_3$

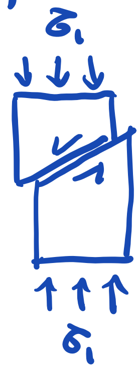
$\Rightarrow$ Two additional circles for the σ_n & τ on those planes





$\Rightarrow$ clearly planes parallel to $\underline{e}_2$ have largest τ!

Failure criteria for shear fracture



Shear fracture is most common type of <u>brittle</u> failure.

Empirical criterion that allows prediction of shear failure.

I) Tresca criterion

Fracture occurs when max. shear stress $\tau_{max} = \tau_{13}$ reaches the shear strength σ_y

$$\boxed{|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \sigma_y}$$

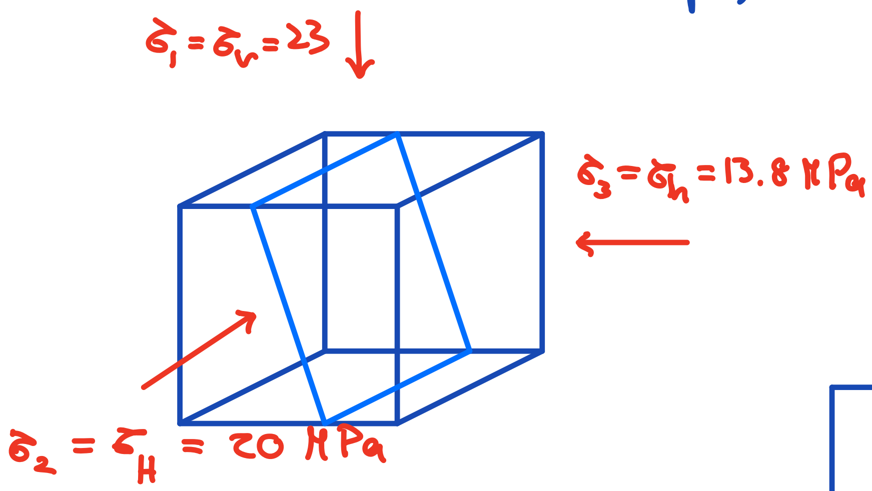
Note: Failure is not affected by intermediate principal stress and mean stress!

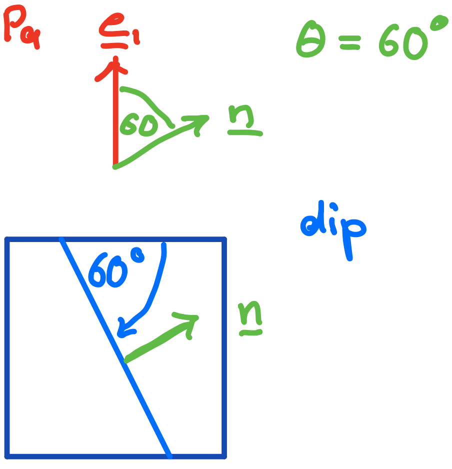
Failure occurs on planes 45° to $\underline{n}_1$

Experiments show angle is smaller than 45°.

Example: Normal & shear stress on a fault

(from N. Esinoza)





Really just 2D problem

mean stress: $\bar{\sigma}_{13} = \frac{23 + 13.8}{2} = 18.4$ MPa

differential stress: $\Delta\sigma_{13} = \frac{23 - 13.8}{2} = 4.6$ MPa (half!)

normal stress: $\sigma_n = \bar{\sigma}_{13} + \Delta\sigma_{13} \cos(2 \cdot 60°) = \underline{16.1 \text{ MPa}}$

shear stress: $\tau = \Delta\sigma_{13} \sin(2 \cdot 60°) = \underline{4.0 \text{ MPa}}$

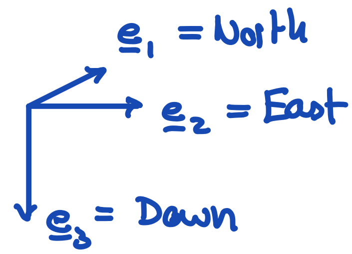
Draw Mohr circle

General tensorial approach

The last approch worked (easily) because the fault was parallel to a principal direction of the stress tensor.

In general we have two (right handed) frames:

1) Geographic frame $\{\underline{e}_i\}$
2) Principal frame of stress tensor $\{\underline{e}_i'\}$



Geographic frame: N-E-D

$$\underline{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \underline{e}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \underline{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Principal directions in geographic frame:

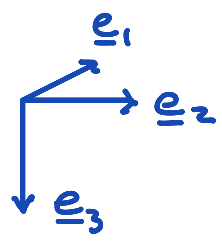
$\underline{e}_1' = \underline{e}_3$ principal stress is vertical

$\underline{e}_3' = \underline{e}_2$ minimal horizontal stress is E-W

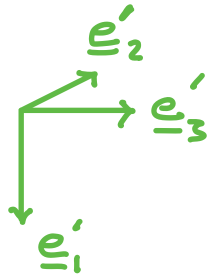
$\underline{e}_2' = \underline{e}_3' \times \underline{e}_1' = \underline{e}_1$ generates right-handed frame

Compare two reference frames

Geographic



Principal dir.



Stress tensor in principal frame:

$$[\underline{\underline{\sigma}}]' = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} = \begin{bmatrix} 23 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 13.8 \end{bmatrix} \text{ MPa}$$

To compute normal & shear stress on fault we need stress in geographic frame $\{\underline{e}_i\}$:

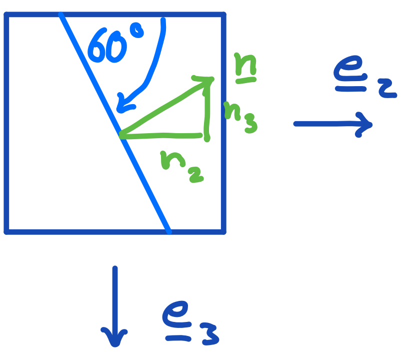
$\Rightarrow$ change of basis tensor: $A_{ij} = \underline{e}_i \cdot \underline{e}_j'$

$$[\underline{\underline{A}}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Stress tensor in geographic frame:

$$[\underline{\underline{\sigma}}] = [\underline{\underline{A}}][\underline{\underline{\sigma}}]'[\underline{\underline{A}}]^T = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 13.8 & 0 \\ 0 & 0 & 23 \end{bmatrix} \text{ MPa}$$

To compute traction we need the normal to the fault:



$$\underline{n} = n_1 \underline{e}_1 + n_2 \underline{e}_2 + n_3 \underline{e}_3$$

$$n_1 = 0$$

$$n_2 = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$n_3 = -\sin\left(\frac{\pi}{2} - \theta\right)$$

traction on fault: $\underline{t}_n = \underline{\underline{\sigma}} \cdot \underline{n} = \begin{bmatrix} 0 \\ 12 \\ -11.5 \end{bmatrix}$ MPa

normal stress: $\sigma_n = \underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n} = 16.1$ MPa ✓

shear stress: $\tau = \sqrt{|\underline{t}_n|^2 + \sigma_n^2} = 4.0$ MPa ✓

II) Coulomb criterion

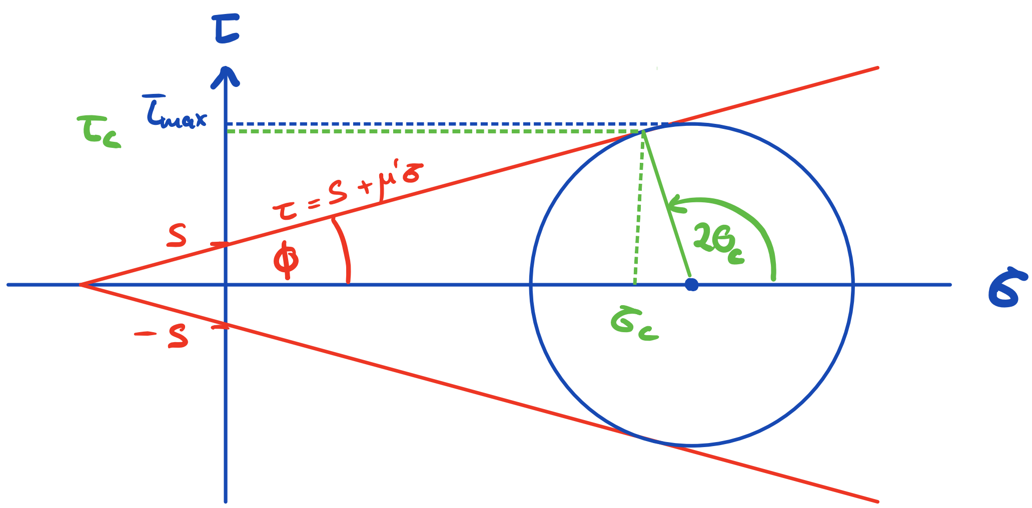
Fracture depends on both mag. of shear stress and the normal stress.

$$\boxed{|\tau| = S + \mu' \sigma}$$

S = cohesive strength ~ 10 – 100 MPa

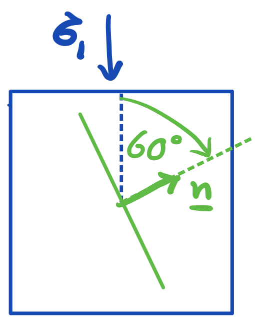
$\mu' = \tan\phi$ internal friction ~ 0.6

$\phi \approx 30°$ angle of internal friction



failure occurs at $\tau_c < \tau_{max}$

angle of failure:



$$\phi + \frac{\pi}{2} + (\pi - 2\theta_c) = \pi$$

$$\theta_c = \underbrace{\frac{\pi}{4} + \frac{\phi}{2}}_{45° + 15°} \approx 60°$$

Byerlee's law

Most brittle rocks already contain pre-existing fractures and fail by reactivating them

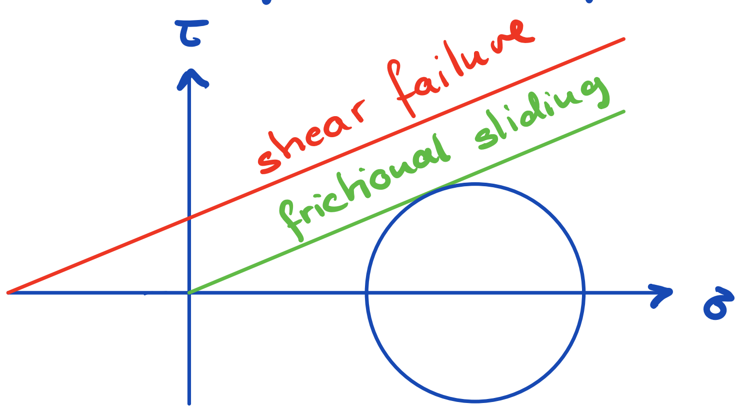
$\Rightarrow$ fail by friction

Criterion for frictional sliding

$$\boxed{|\tau| = S_0 + \mu_0 \sigma}$$

S_0 = cohesion of fault $\sim$ 1 - 10 MPa

μ_0 = coefficient of friction $\sim$ 0.5 - 0.8



$S_0 \ll S$

Strength of brittle rocks is determined by frictional sliding.