Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density uniform: $\rho_0(X) = \rho_0$

2) Fluid is incompressible $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress field is Newtonian

$$\underline{\underline{\sigma}} = -p \underline{\underline{I}} + \mathbb{C} \nabla_x \underline{v}$$

where $p(x,t)$ is pressure field

and $\mathbb{C}$ is a fourth-order tensor field

with left minor symmetry $(\mathbb{C}\underline{\underline{A}})^T = \mathbb{C}\underline{\underline{A}}$

$\Rightarrow$ ensures the symmetry $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \Rightarrow$ ang. mom.

and trace condition $\mathrm{tr}\, \mathbb{C}\underline{\underline{A}} = 0$ if $\mathrm{tr}\, \underline{\underline{A}} = 0$

$\Rightarrow$ $p = \frac{1}{3} \mathrm{tr}(\underline{\underline{\sigma}})$ when $\mathrm{tr}(\nabla_x \underline{v}) = \nabla_x \cdot \underline{v} = 0$

Prop 1 + 2 $\Rightarrow$ $p(\underline{x},t) = \rho_0 > 0$

Reactive stress: $\underline{\underline{\sigma}}^r = -p \underline{\underline{I}}$

p is multiplier for $\nabla_x \cdot \underline{v} = 0$

Active stress: $\underline{\underline{\sigma}}^a = \mathbb{C} \nabla_x \underline{v} = 2\mu\, \mathrm{sym}(\nabla_x \underline{v})$

by frame indifference

$\mu =$ absolute viscosity

In limit $\mu \to 0$ Newtonian fluid reduces to ideal fluid.

Navier - Stokes Equations

Setting $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \, \mathrm{sym}(\nabla_x \underline{v})$

we obtain lin. mom. balance

$$\rho_0 \dot{v} = \nabla_x \cdot \left(-p\underline{\underline{I}} + 2\mu \, \mathrm{sym}(\nabla_x \underline{v})\right) + \rho_0 \underline{b}$$

from mat. deriv. $\dot{\underline{v}} = \frac{\partial v}{\partial t} + (\nabla_x \underline{v})\underline{v}$

assuming $\mu =$ constant we have

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x \cdot \nabla_x \underline{v} + \mu \nabla_x \cdot (\nabla_x \underline{v})^T$$

$$\nabla_x \cdot \nabla_x \underline{v} = v_{i,jj} \, \underline{e}_i = \nabla_x^2 \underline{v}$$

$$\nabla_x \cdot (\nabla_x \underline{v})^T = v_{j,ij} \, \underline{e}_i = v_{j,ji} \, \underline{e}_i = \nabla_x (\cancel{\nabla_x \cdot \underline{v}}^{\,0})$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x^2 \underline{v}$$

so that

$$\boxed{\begin{gathered} \rho_0 \left[\frac{\partial v}{\partial t} + (\nabla_x \underline{v})\underline{v}\right] = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho_0 b \\ \nabla_x \cdot \underline{v} = 0 \end{gathered}}$$

Frame-Indifference of Newtonian fluid model

We already showed the indifference of constraint.

$\Rightarrow$ focus on active stress

$$\underline{\underline{\sigma}}^a = \mathbb{C}\, \nabla_x \underline{v} = 2\mu\, \mathrm{sym}(\nabla_x \underline{v}) = 2\mu \underline{\underline{d}}$$

Check left minor symmetry of $\mathbb{C}$

$$(\mathbb{C}\, \nabla_x \underline{v})^T = (2\mu\, \underline{\underline{d}})^T = 2\mu\, \underline{\underline{d}}^T = 2\mu \underline{\underline{d}} = \mathbb{C}\, \underline{\underline{\nabla_x v}} \quad \checkmark$$

Check trace condition

$$\mathrm{tr}(\mathbb{C}\, \nabla_x \underline{v}) = 2\mu\, \mathrm{tr}(\underline{\underline{d}}) = 0 \quad \text{if} \quad \mathrm{tr}(\underline{\underline{d}}) = 0$$

Now assume a superposed rigid motion

$$\underline{x}^* = \underline{\underline{Q}}(t)\, \underline{x} + \underline{c}(t)$$

where $\quad \underline{\underline{\sigma}}^{a*}(\underline{x}^*, t) = \underline{\underline{\sigma}}^{a*} \quad$ and $\quad \underline{\underline{d}}^* = \underline{\underline{d}}^*(\underline{x}^*, t)$

$$\underline{\underline{\sigma}}^{a*} = 2\mu\, \underline{\underline{d}}^* = 2\mu\, \underbrace{\underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^*}_{\underline{\underline{d}}^*} = \underline{\underline{Q}}(2\mu \underline{\underline{d}})\underline{\underline{Q}}^T = \underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^T \quad \checkmark$$

see also discussion in Lecture 20

Mechanical energy considerations

Stress power of Newtonian fluid is

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-pI + 2\mu \underline{\underline{d}}) : \underline{\underline{d}} = -p \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla_x \cdot \underline{v} = 0} + 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

$$= 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

From reduced Clausius-Duhem inequality

$$\rho \dot{\psi} \leq 2\mu \underbrace{\underline{\underline{d}} : \underline{\underline{d}}}_{\geq 0}$$

$\Rightarrow$ only if $\mu > 0$ energy is dissipated during the flow $\dot{\psi} < 0$

Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in Ideal and Newtonian fluids.

First some useful results:

1) Integration by parts in fixed domain Ω with "no slip" boundaries $\underline{v} = \underline{0}$ on $\partial\Omega$.

$$\boxed{\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v}\, dV_x = -\int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v})\, dV_x}$$

To see this consider $(v_{i,j} v_i)_{,j} = v_{i,jj} v_i + v_{i,j} v_{i,j}$

$$(\nabla_x^2 \underline{v})\cdot \underline{v} = v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$
$$= \nabla\cdot((\nabla_x \underline{v})^T \underline{v}) - (\nabla_x \underline{v}):(\nabla_x \underline{v})$$

substituting into integral and applying div-thm

$$\int_\Omega (\nabla_x^2 \underline{v})\cdot \underline{v}\, dV_x = \int_{\partial\Omega} (\nabla_x v)^T \cancelto{0}{\underline{v}} \cdot \underline{n}\, dA_x - \int_\Omega (\nabla_x \underline{v}):(\nabla_x \underline{v})\, dV_x$$

2) Poincaré Inequality

$$\|\underline{u}\|_\Omega \le \lambda \|\nabla_x u\|_\Omega \quad \text{for} \quad \underline{u} = 0 \quad \partial\Omega \quad \lambda > 0$$

using standard inner product

$$\boxed{\int_\Omega |\underline{u}|^2\, dV_x \le \lambda \int_\Omega \nabla_x \underline{u} : \nabla_x \underline{u}\, dV_x}$$

Notice λ has units of L^2 and scales with area of Ω.

Kinetic Energy of Newtonian & Ideal fluids

Consider a fixed domain Ω with $\underline{v} = 0$ on $\partial\Omega$

and a conservative body force $b = -\nabla_x \Phi$.

The kinetic energy is given by

$$K(t) = \int_\Omega \frac{1}{2}\rho_0 |\underline{v}|^2 \, dV_x \quad \text{and} \quad K(0) = K_0$$

I) Newtonian fluid

$$\boxed{K(t) \leq e^{-2\mu t/\lambda\rho_0} K_0}$$

The kinetic energy of a Newtonian fluid dissipates to zero exponentially fast.

II) Ideal fluid

$$\boxed{K(t) = K_0}$$

The kinetic energy of ideal fluid is constant.

By def. of K we have

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho_0 \frac{d}{dt}|\underline{v}|^2 \, dV_x = \int_\Omega \rho_0 \, \dot{\underline{v}} \cdot \underline{v} \, dV_x$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \Psi$

$$\frac{d}{dt}K(t) = \int_\Omega (\mu \nabla_x^2 \underline{v} - \nabla \Psi) \cdot \underline{v} \, dV_x$$

show $\int_\Omega \nabla_x \psi \cdot \underline{v} \, dV_x = 0$

$$\nabla_x \cdot (\varphi \, \underline{v}) = \nabla_x \varphi \cdot \underline{v} + \underbrace{\cancel{(\nabla_x \cdot \underline{v})}}_{0} \varphi = \nabla_x \varphi \cdot \underline{v}$$

substitute and use Div-Thm

$$\frac{d}{dt}K(t) = \int_\Omega \mu (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x - \int_{\partial\Omega} \psi \underbrace{\cancel{\underline{v}}}_{0} \cdot \underline{n} \, dA_x$$

using integration by parts

$$\frac{d}{dt}K(t) = -\mu \int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x$$

for Ideal fluid $\mu = 0 \Rightarrow K(t) = K_0$

for Newtonian fluid apply Poincare inequality

$$\frac{d}{dt}K(t) \leq -\frac{\mu}{\lambda} \int_\Omega |\underline{v}|^2 dV_x = -\frac{2\mu}{\lambda \rho_0} K(t)$$

so that we have

$$\boxed{\frac{d}{dt}K(t) \leq -\frac{2\mu}{\lambda \rho_0} K(t)}$$

where λ depends on area of the domain.

Solve by seperation of parts

$$\frac{dk}{k} \leq -\frac{2\mu}{\rho_0 \lambda} dt = -\alpha dt$$

$$\ln k \leq -\alpha t + c_0$$

$$k \leq c_1 e^{-\alpha t}$$

Initial condition $\quad k(0) \leq c_1 = k_0$

$$\Rightarrow \quad k(t) \leq k_0 e^{-\frac{2\mu}{\lambda \rho_0} t} \quad ✓$$

In absence of fluid motion on the boundary fluid motion decays exponentially.

The rate of decay depends

$$\boxed{\nu = \frac{\mu}{\rho_0}} \quad \text{kinematic viscosity}$$