Scaling Navier Stokes Equations

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v} = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho g$$

reduced pressure:

$$-\nabla_x p + \rho g = -\nabla_x p - \rho g \hat{z} = -\nabla(p + \rho g z) = -\nabla \pi$$

we have

$$\rho_0 \left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v} \right) - \mu \nabla_x^2 \underline{v} = -\nabla_x \pi$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters.

- Dependent variables: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parametess: $\rho \left[\frac{M}{L^3}\right]$ $\quad \mu \left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\mu}{\rho} \left[\frac{L^2}{T}\right]$

  + Geometry, BC, IC

Use parameters to scale the variables:

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

substitute into governing equations

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c}(\nabla_x' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla_x' \pi'$$

Option 1: Scale to accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1}(\nabla_x' \underline{v}')\underline{v}' - \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2}\nabla_x'^2 \underline{v}' = -\underbrace{\frac{\pi_c t_c}{x_c \rho_0 v_c}}_{\Pi_3}\nabla_x' \pi'$$

where $\nu = \frac{\mu}{\rho}$ "momentum diffusivity"

Three dimensionless groups $\Rightarrow$ define time scale

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective scale $t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1 \Rightarrow$ diffusive scale $t_c = t_D = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale

$\Pi_3 = \frac{\pi_c t_c}{x_c \rho_0 v_c} = 1 \Rightarrow \pi_c = \frac{x_c \rho_0 v_c}{t_c}$

Choose a diffusive time scale $t_c = \frac{x_c^2}{\nu}$

$$\frac{\partial \underline{v}}{\partial t} + \frac{v_c x_c}{\nu}(\nabla_x' \underline{v}')\underline{v}' - \nabla_x'^2 \underline{v} = -\nabla_x' \pi'$$

$\Rightarrow$ one remaining dim. less group

$\boxed{Pe_m = \frac{t_D}{t_A} = \frac{v_c x_c}{\nu} = Re}$ Reynolds number

Hence we have

$$\frac{\partial \underline{v}'}{\partial t'} + Re\,(\nabla'_x \underline{v}')\underline{v}' - \nabla'^2_x \underline{v} = -\nabla'_x \pi'$$

Advective momentum transport vanishes as $Re \to 0$

For viscous flow of glaciers:

$\rho_0 = 10^3 \frac{kg}{m^3}$ $\quad v_c = 100 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\mu = 10^{14}$ Pas $\quad x_c = 10^2$ m (thickness)

$$Re = \frac{v_c x_c \rho_0}{\mu_c} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

$\Rightarrow$ advective momentum transport is negligible

Momentum balance simplifies

$$\boxed{\frac{\partial \underline{v}'}{\partial t} - \nabla'^2_x \underline{v} = -\nabla'_x \pi' \quad \& \quad \nabla_x \cdot \underline{v} = 0}$$ linear !

But is it worth resolving diffusive timescale ?

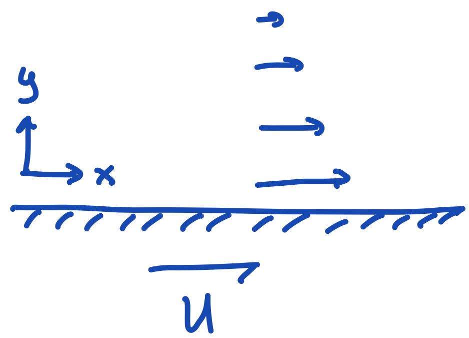
$$t_D = \frac{x_c^2 \rho_0}{\mu} = 10^{4+3-14}\,s = 10^{-7}\,s$$

This is very short compared to 100 years of glacier response. Not worth resolving transients.

Can't eliminate transient term because we scaled to it $\Rightarrow$ scale to diffusion term.

Rayleigh's problem

- Semi-infinite half-space
- Stationary fluid
- Impulsively started plate with velocity U.



$$v_c = U \quad \rightarrow \quad Re = \frac{U x_c \rho_0}{\mu} \ll 1 \quad \Rightarrow \quad U \ll \frac{\mu}{x_c \rho_0}$$

But what is x_c? Not obvious

Redimensionalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} u \\ w \end{pmatrix}$$

Simplify the equations:

Domain is infinite in x but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

Flow is horizontal: $\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \Rightarrow w = 0$

From continuity: $\frac{\partial u}{\partial x} + \cancel{\frac{\partial w}{\partial y}}^{\,0} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$

$\nabla^2 \underline{v} = v_{i,jj}\, \underline{e}_i \qquad i,j \in \{1,2\}$

$$= \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancel{u_{xx}}^{\,0} & u_{yy} \\ \cancel{w_{xx}}^{\,0} & \cancel{w_{yy}}^{\,0} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting we have

x-mom.: $\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0$

y-mom.: $0 = -\frac{\partial \pi}{\partial y}$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}} \quad \text{with} \quad u(0,y) = 0 \quad u(t,0) = U$$

This is identical to heating a semi-infinite rod from the end.

Problem has self-similar solution in

$$\eta = \frac{y}{\sqrt{4\nu t}} \quad \text{and} \quad u(y,t) = U f(\eta)$$

where $\sqrt{4\nu t}$ takes role of char. length that depends on t.

derivatives: $\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$

The derivatives of u transform as:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$$

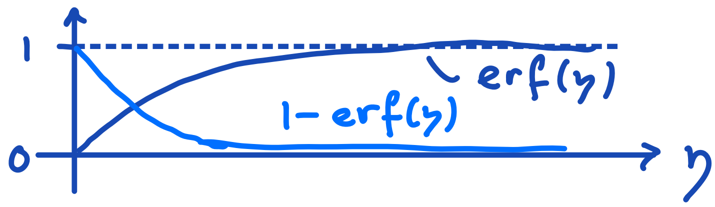
substituting into PDE:

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \qquad \text{with} \quad f(\eta=0) = 1$$

Reduce PDE in y and t to ODE in η

Solution: $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)

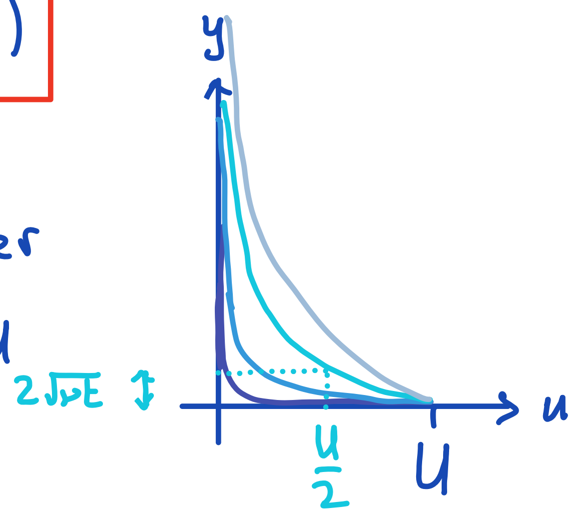
where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi$ error function



Resubstituting for $f = \frac{u}{U}$ and $\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4Dt}}\right)\right)}$$

Diffusive boundary layer where momentum added by boundary penetrates into the quiescent fluid.



$\nu = \frac{\mu}{\rho_0}$ is Diffusion coefficient.

Stokes Equation

Scaling to mom. diffusion

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho_0 v_c^2}{x_c}(\nabla_x' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla_x' \pi'$$

divide by $\mu v_c / x_c^2$

$$\frac{x_c^2}{\nu t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{v_c x_c}{\nu}(\nabla_x' \underline{v}')\underline{v}' - \nabla_x'^2 \underline{v} = -\underbrace{\frac{\pi_c x_c}{\mu v_c}}_{1}\nabla_x' \pi'$$

choose $t_c = t_A = \frac{x_c}{v_c}$ $\quad 1 \Rightarrow \pi_c = \frac{\mu v_c}{x_c}$

$$Re\left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}\right) - \nabla_x'^2 \underline{v} = -\nabla_x' \pi'$$

In the limit $Re \ll 1$ we obtain

$$\boxed{\begin{aligned} \nabla_x'^2 \underline{v}' &= \nabla_x' \pi' \\ \nabla_x' : \underline{v}' &= 0 \end{aligned}}$$

Stokes equations dimension less

Redimensionalize : $\underline{v}' = \frac{\underline{v}}{v_c}$ $\quad \pi' = \frac{\pi}{\frac{\mu v_c}{x_c}}$ $\quad x' = \frac{x}{x_c}$

$$\frac{x_c^2}{v_c}\nabla_x^2 \underline{v} = \frac{x_c^2}{\mu v_c}\nabla_x \pi$$

$$\boxed{\begin{aligned} \mu \nabla_x^2 \underline{v} &= \nabla_x \pi \\ \nabla_x : \underline{v} &= 0 \end{aligned}}$$

Dimensional Stokes equation

Properties of the Stokes Equation

1) Linearity

Construct solutions by linear superposition

2) Instanteneity

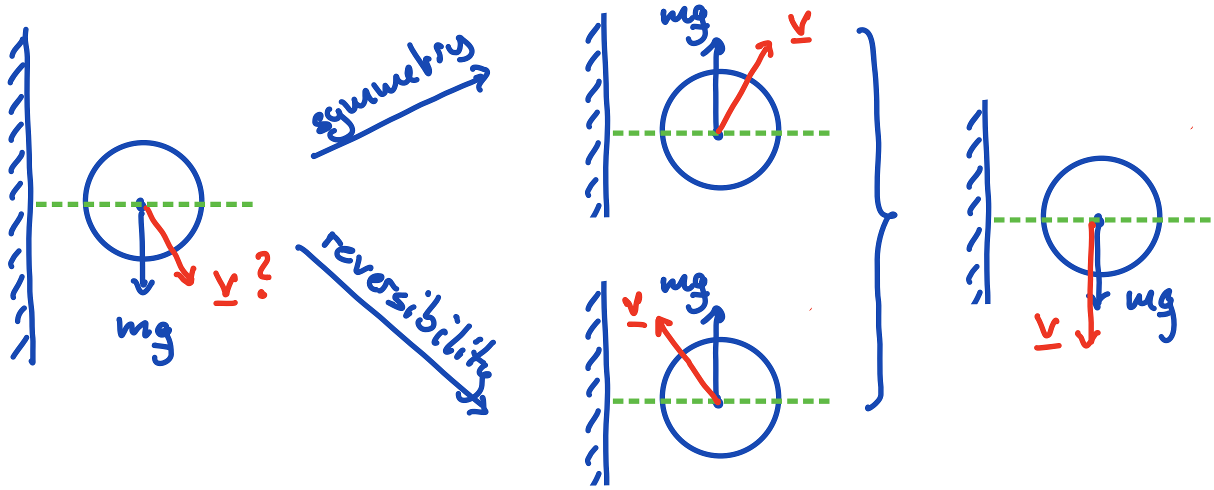
No time dependence other than due to time varying boundary conditions

3) Reversibility

If the body force and the velocity on boundary are reversed so is the velocity everywhere.

These tell us a lot about possible solutions.

Example: Sphere falling next to a wall



Helmholtz minimum dissipation Theorem

For a given domain and boundary conditions the rate of dissipation in a Stokes flow is less or equal to any other incompressible flow.

Stokes flow : $\underline{v}$ $\underline{\underline{\sigma}}$ $\qquad \nabla \cdot \underline{v} = 0$

Other flow : $\underline{v}'$ $\underline{\underline{\sigma}}'$ $\underline{\underline{d}}'$ $\qquad \nabla \cdot \underline{v} = 0$

Dissipation :

$$\mathcal{D} = 2\mu \int \underline{\underline{d}} : \underline{\underline{d}} \, dV$$

$$\mathcal{D}' = 2\mu \int \underline{\underline{d}}' : \underline{\underline{d}}' \, dV = 2\mu \int \underline{\underline{d}} : \underline{\underline{d}} + (\underline{\underline{d}} - \underline{\underline{d}}') : (\underline{\underline{d}} - \underline{\underline{d}}') \, dV$$

$$d'd' - d'd - dd' + dd$$