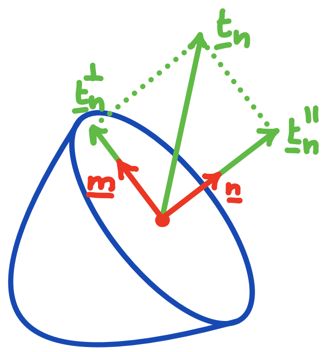
Normal and Shear Stresses



Consider an arbitrary surface in B with normal $\underline{n}$. Then we have the two projection matrices

$$\underline{\underline{P}}^\parallel = \underline{n} \otimes \underline{n} \quad \text{and} \quad \underline{\underline{P}}^\perp = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

that define the

normal stress: $\underline{t}_n^\parallel = \underline{\underline{P}}^\parallel \underline{t}_n = (\underline{n} \cdot \underline{t}_n)\, \underline{n} = \sigma_n \underline{n}$

shear stress: $\underline{t}_n^\perp = \underline{\underline{P}}^\perp \underline{t}_n = (\underline{m} \cdot \underline{t}_n)\, \underline{m} = \tau\, \underline{m}$

The magnitudes of these stresses are

$$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} \quad \text{or} \quad \sigma_n = n_i \sigma_{ij} n_j$$

$$\tau = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \sigma\, \underline{n} \quad \text{or} \quad \tau = m_i \sigma_{ij} n_j$$

If $\sigma_n > 0$ the normal stresses are tensile if $\sigma_n < 0$ the normal stresses are compressive.

From geometry: $\underline{t}_n = \underline{t}_n^\parallel + \underline{t}_n^\perp$

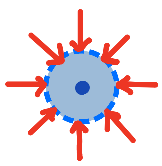
$$|\underline{t}_n|^2 = |\sigma_n \underline{n}|^2 + |\tau\, \underline{n}|^2 = \sigma_n^2 + \tau_n^2$$

Simple states of stress

I) Hydrostatic stress

$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{n}$ for all $\underline{n}$

$\underline{t}_n^{\parallel} = \underline{\underline{P}}_n^{\parallel}\,\underline{t} = (\underline{n}\otimes\underline{n})(-p\,\underline{n}) = -p\,(\underline{n}\cdot\underline{n})\,\underline{n} = -p\,\underline{n}$



$\Rightarrow \underline{t}_n = \underline{t}_n^{\parallel} \qquad \underline{t}_n^{\perp} = 0$

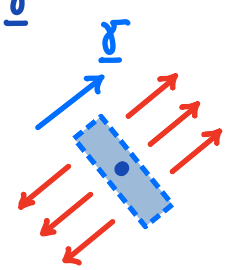
all normal stresses are $\sigma_1 = \sigma_2 = \sigma_3 = -p$

no shear stresses on any plane

II) Uniaxial stress

$\boxed{\underline{\underline{\sigma}} = \sigma\,\underline{\gamma}\otimes\underline{\gamma}} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \sigma\,(\underline{\gamma}\cdot\underline{n})\,\underline{\gamma}$

($\underline{\gamma}$ is unit vector) Traction is always parallel to $\underline{\gamma}$



and vanishes on surfaces with

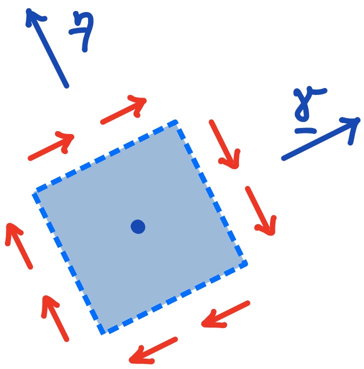
normal perpendicular to $\underline{\gamma}$.

$\sigma > 0$: pure tension

$\sigma < 0$: pure compression

III, Pure shear stress $\quad \underline{\gamma} \cdot \underline{\eta} = 0$

$$\boxed{\underline{\underline{\sigma}} = \tau \, (\underline{\gamma} \otimes \underline{\eta} + \underline{\eta} \otimes \underline{\gamma})} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\, \underline{n} = \tau (\underline{\eta} \cdot \underline{n}) \underline{\gamma} + \tau (\underline{\gamma} \cdot \underline{n}) \underline{\eta}$$



$\underline{n} = \underline{\eta} : \quad \underline{t}_n = \tau \underline{\gamma}$

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IV, Plane stress

If there exists a pair of orthogonal vectors $\underline{\gamma}$ and $\underline{\eta}$ such that the matrix representation of $\underline{\underline{\sigma}}$ in the frame $\{\underline{\gamma}, \underline{\eta}, \underline{\gamma} \times \underline{\eta}\}$ is of the form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then a state of plane stress exists.

Spherical and deviatoric stress tensors

The Cauchy stress tensor can be decomposed as $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_S + \underline{\underline{\sigma}}_D$

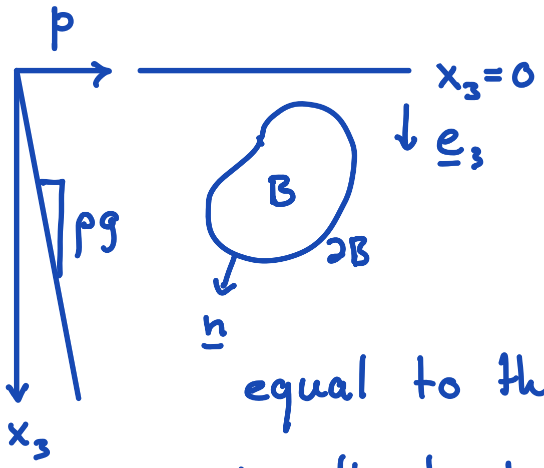
spherical stress tensor: $\underline{\underline{\sigma}}_S = -p\underline{\underline{I}}$ $\quad p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})$

deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} + p\underline{\underline{I}}$

The pressure $p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ can be interpreted as the mean normal traction. The spherical stress is the part of $\underline{\underline{\sigma}}$ that changes the volume of the body. Note that $p > 0$ corresponds to compression.

The deviatoric stress is the part of $\underline{\underline{\sigma}}$ that changes the shape of a body without changing its volume. By definition $\mathrm{tr}\,\underline{\underline{\sigma}}_D = 0$.

Example: Archimedes' principle



Any object, wholly or partially submerged in a fluid is bouyed up by a force equal to the weight of the fluid displaced by the body.

Q: Is the buoyancy force a body or a surface force?

Hydrostatic pressure acts on the boundary of the object. $\Rightarrow$ external surface force

Buoyancy force $\rightarrow$ resultant surface force

$$\boxed{\underline{r}_s[\partial B] = -W\underline{e}_3 = -\rho g V_B \underline{e}_3}$$

ρ = water density

Hydrostatic pressure: $p = \rho g x_3$

Hydrostatic traction on ∂B: $\underline{t} = -p\underline{n}$

Resulting surface force:

$$\underline{r}_s[\partial B] = \int_{\partial B} \underline{t}\, dA = -\int_{\partial B} p\, \underline{n}\, dA$$

need to convert this to volume integral

$\Rightarrow$ Divergence Thm $\int_\Omega \nabla \cdot \underline{f}\, dV = \int_{\partial\Omega} \underline{f} \cdot \underline{n}\, dA$

Multiply by arbitrary const. vector

$$\underline{c} \cdot \underline{r}_s[\partial B] = -\underline{c} \cdot \int_{\partial B} p\, \underline{n}\, dA = -\int_{\partial B} \underline{c} \cdot (p\underline{n})\, dA = -\int_{\partial B} \underbrace{(p\underline{c})}_{\underline{f}} \cdot \underline{n}\, dA$$

$$= -\int_B \nabla \cdot (p\underline{c})\, dV$$

$$= -\int_B \underline{c} \cdot \nabla p + p \cancel{\nabla \cdot \underline{c}}^{\,0}\, dV = -\underline{c} \cdot \int_B \nabla p\, dV$$

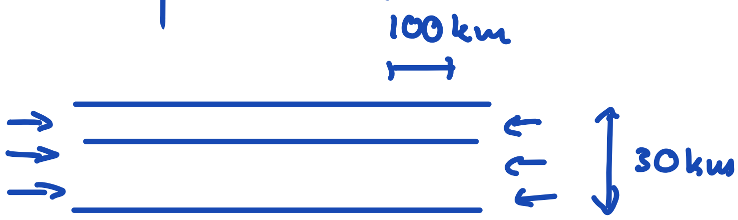
$$\Rightarrow \quad \underline{r}_s[\partial B] = -\int_{\partial B} p\, \underline{n}\, dA = -\int_B \nabla p\, dV$$

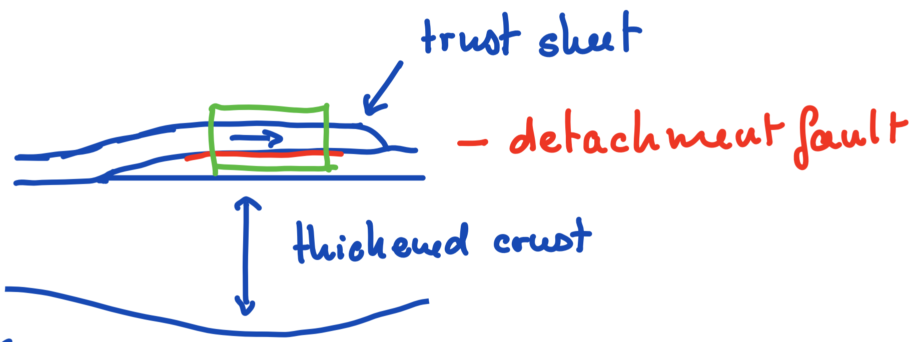
Note: Gradient thm $\boxed{\int_{\partial\Omega} \phi\, \underline{n}\, dA = \int_\Omega \nabla \phi\, dV}$

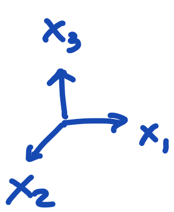
$$\underline{r}_s[\partial B] = -\int_B \nabla p\, dV \qquad \nabla p = \nabla(\rho g x_3) = \rho g\, \underline{e}_3$$

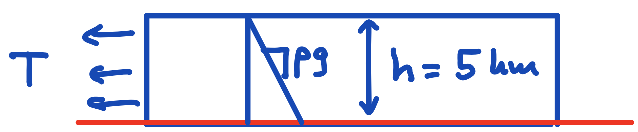
$$\underline{F}_s[\partial B] = -\int_B \rho g\, \underline{e}_3\, dV = -\rho g\, \underline{e}_3 \underbrace{\int_B dV}_{V_B} = -\rho g V_B\, \underline{e}_3 \quad ✓$$

Example: Fault block on detachment









Normal stresses:

Verital stress: $\sigma_{33} = \rho g h$

Horizontal stress (x_1-dir): $\sigma_{11} = \kappa\, \sigma_{33} - T$

Horizontal stress (x_2 dir): $\sigma_{22} = \kappa\, \sigma_{33}$

In fluid $\kappa = 1$, but in rock $\kappa < 1$ due to finite strength.

T is tensile tectonic stress

Assume only shear stress is in 1-3 coord. plane

$\sigma_{13} = \sigma_{31} = \mu (\rho g h)$ $\quad$ μ = coefficient of friction

$\sigma_{21} = \sigma_{12} = 0$ $\quad$ $\sigma_{23} = \sigma_{32} = 0$

This results in following stress tensor:

$$\underline{\underline{\sigma}} = \begin{bmatrix} \kappa\rho g h - T & 0 & \mu \rho g h \\ 0 & \kappa \rho g h & 0 \\ \mu \rho g h & 0 & \rho g h \end{bmatrix}$$

Traction on basal plane:

$$\underline{t}(\underline{e}_3) = \underline{\underline{\sigma}}\, \underline{e}_3 = \begin{bmatrix} \mu \\ 0 \\ 1 \end{bmatrix} \rho g h$$

Normal stress of fault: $\underline{t}(\underline{e}_3) \cdot \underline{e}_3 = \rho g h$

Shear stress on fault: $\underline{t}(\underline{e}_3) \cdot \underline{e}_1 = \mu \rho g h$

Assume following numbers:

$\rho = 2700 \ kg/m^3$ $\qquad h = 5000 \ m$

$g = 9.8 \ m/s^2$ $\qquad T = 50 \ MPa$

$K = 0.3$ $\qquad \mu = 0.6$

$$\Rightarrow \quad \underline{\underline{\sigma}} = \begin{bmatrix} -10.3 & 0 & 79.4 \\ 0 & 39.7 & 0 \\ 79.4 & 0 & 132.4 \end{bmatrix} \ MPa$$