Tensor algebra in components

Addition : $\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$

$$H_{ij}\, \underline{e}_i \otimes \underline{e}_j = S_{ij}\, \underline{e}_i \otimes \underline{e}_j + T_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

$$= (S_{ij} + T_{ij})\, \underline{e}_i \otimes \underline{e}_j$$

$$\boxed{H_{ij} = S_{ij} + T_{ij}}$$

Scalar multiplication : $\underline{\underline{H}} = \alpha \underline{\underline{S}} \Rightarrow \boxed{H_{ij} = \alpha S_{ij}}$

Product : $\underline{\underline{H}} = \underline{\underline{S}}\, \underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij} (\underline{e}_i \otimes \underline{e}_j)\, T_{kl} (\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} \underbrace{(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)}_{\text{product of two dyads}}$$

$$= S_{ij} T_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}}\, \underline{e}_i \otimes \underline{e}_l$$

$$= S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$H_{il}\, \underline{e}_i \otimes \underline{e}_l = S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\Rightarrow \boxed{H_{il} = S_{ij} T_{jl}}$$

note the dummy j !

Transpose of a tensor

To any $\underline{\underline{S}} \in \mathcal{V}^2$ we associate a transpose $\underline{\underline{S}}^T \in \mathcal{V}^2$ the unique tensor such that

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}} \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

This implies that $S^T_{ij} = S_{ji}$ as follows

$$(S_{ij} u_j \underline{e}_i) \cdot (v_\ell \underline{e}_\ell) = (u_k \underline{e}_k) \cdot (S^T_{ij} v_j \underline{e}_i)$$

$$S_{ij} u_j v_\ell (\underline{e}_i \cdot \underline{e}_\ell) = S^T_{ij} v_j u_k (\underline{e}_k \cdot \underline{e}_i)$$

$$S_{ij} u_j v_\ell \, \delta_{i\ell} = S^T_{ij} v_j u_k \, \delta_{ki}$$

$$S_{ij} u_j v_i = S^T_{ij} v_j u_i \qquad \text{rename indices } i \leftrightarrow j \text{ on rhs}$$

$$S_{ij} u_j v_i = S^T_{ji} u_j v_i$$

$$\Rightarrow \quad S_{ij} = S^T_{ji} \quad \checkmark$$

Properties of transpose:

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew-symmetric if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Symmetric-Skew decomposition:

Any tensor $\underline{\underline{S}} \in \mathcal{V}^2$ can be written as

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}} \qquad \begin{aligned} \underline{\underline{E}} &= \underline{\underline{E}}^T \\ \underline{\underline{W}} &= -\underline{\underline{W}}^T \end{aligned}$$

Note: $\underline{\underline{W}} = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$ only 3 indep. comp.

$\Rightarrow$ can be related to an axial vector $\underline{w}$

$$\boxed{\underline{\underline{W}}\,\underline{v} = \underline{w} \times \underline{v}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

Relation :

note jk often flipped !!

$$W_{ij} = -\epsilon_{ijk} w_k$$

$$w_k = -\tfrac{1}{2} \epsilon_{ijk} W_{ij}$$

$$\underline{\underline{W}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Trace of a tensor

We define the trace of a dyad as

$$\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i$$

this implies that

$$\boxed{\mathrm{tr}(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$$

as follows $\quad \mathrm{tr}(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij}\, \mathrm{tr}(\underline{e}_i \otimes \underline{e}_j)$

$$= A_{ij}\, \delta_{ij} = A_{ii}$$

Properties: $\quad \mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Decomposition: $\quad \boxed{\underline{\underline{A}} = \alpha \underline{\underline{I}} + \mathrm{dev}\, \underline{\underline{A}}}$

Spherical tensor: $\alpha \underline{\underline{I}}$ $\quad$ where $\quad \alpha = \frac{1}{3}\mathrm{tr}(\underline{\underline{A}})$

Deviatoric tensor: $\mathrm{dev}\, \underline{\underline{A}} = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\mathrm{tr}(\mathrm{dev}\, \underline{\underline{A}}) = 0$$

Tensor scalar product (Contraction)

analogous to scalar product of vectors

$$\boxed{\underline{\underline{A}} : \underline{\underline{B}} = \mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}}$$

scalar ∇

explicitly:

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \dots$$
$$A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} + \dots$$
$$A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33}$$

The index expression is derived as follows

$$\underline{\underline{A}} : \underline{\underline{B}} = \mathrm{tr}(\underline{\underline{A}}^T B) =$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = (A_{ji}\, \underline{e}_i \otimes \underline{e}_j)(B_{kl}\, \underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_j \cdot \underline{e}_k)(\underline{e}_i \otimes \underline{e}_l) = A_{ji} B_{kl} \delta_{jk}\, \underline{e}_i \otimes \underline{e}_l$$
$$= A_{ji} B_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ji} B_{jl} \delta_{il} = A_{ji} B_{ji} = A_{ij} B_{ij} \quad \checkmark$$

Properties: 1) $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{B}} : \underline{\underline{A}}$

$$2)\ (\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

First follows from prop. of trace

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = tr((\underline{\underline{A}}^T \underline{\underline{B}})^T) = tr(\underline{\underline{B}}^T \underline{\underline{A}}) = \underline{\underline{B}} : \underline{\underline{A}}$$

Second property $[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$

$$(\underline{a} \otimes b) : (\underline{c} \otimes \underline{d}) = [\underline{a} \otimes \underline{b}]_{ij} [\underline{c} \otimes \underline{d}]_{ij} = a_i b_j c_i d_j$$

$$= a_i c_i b_j d_j$$

$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

A common norm for tensors is

$$\boxed{|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = \sqrt{A_{ij} A_{ij}} \geq 0}$$

Note: Tensor scalar product will be important to express the work done during deformation. For example the shear heating in glaciology.

add sym skew contraction $\Rightarrow$ useful

also important for driv. of tensor functions for $\nabla \times \nabla \phi = 0$

Determinant and Inverse

The determinat of $\underline{\underline{A}} \in \mathcal{V}^2$ is the salar

$$\boxed{\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} [\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}}$$

where $[\underline{\underline{A}}]_{i1}$, $[\underline{\underline{A}}]_{j2}$, $[\underline{\underline{A}}]_{k3}$ are the columns of $[\underline{\underline{A}}]$

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$

$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$

$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}}) \quad (\underline{\underline{A}} \text{ is } n \times n)$

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$.

If $\det \underline{\underline{A}} \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\boxed{\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Add relation to triple scalar product!

Properties: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

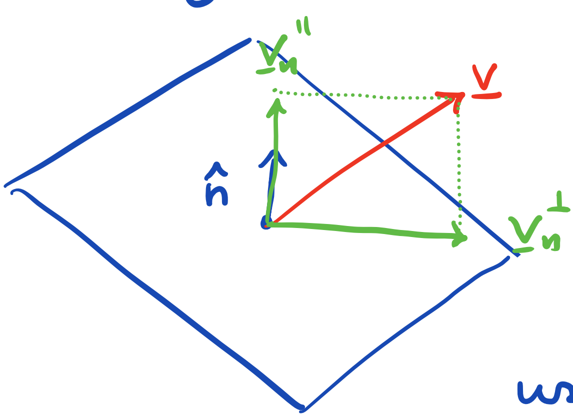
$$(\underline{\underline{A}}^{-1})^{T} = (\underline{\underline{A}}^{T})^{-1}$$

$$(\alpha A)^{-1} = \frac{1}{\alpha}\underline{\underline{A}}^{-1}$$

$$\det(\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}})^{-1} = \frac{1}{\det(\underline{\underline{A}})}$$

Projection & Reflection tensors

commonly used to partition forces on a surface.



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

$$\underline{v}_n^{\parallel} = (\underline{v}\cdot\underline{\hat{n}})\,\underline{\hat{n}}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

use dyadic property

$$\underline{v}_n^{\parallel} = (\underline{v}\cdot\underline{\hat{n}})\underline{\hat{n}} = (\underline{\hat{n}}\otimes\underline{\hat{n}})\,\underline{v} = \underline{\underline{P}}_n^{\parallel}\,\underline{v}$$

$$\underline{v}_n^{\perp} = \underline{v} - (\underline{\hat{n}}\otimes\underline{\hat{n}})\underline{v} = (\underline{\underline{I}} - \underline{\hat{n}}\otimes\underline{\hat{n}})\,\underline{v} = \underline{\underline{P}}_n^{\perp}\,\underline{v}$$

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \underline{\hat{n}}\otimes\underline{\hat{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \underline{\hat{n}}\otimes\underline{\hat{n}}}$$

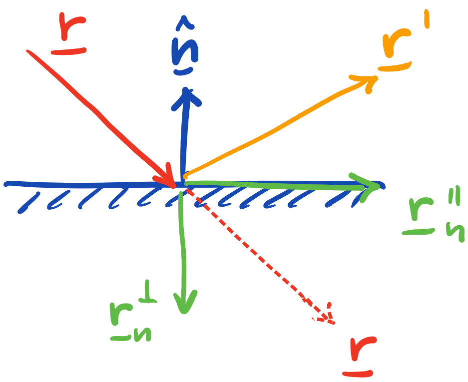
Properties: $\underline{\underline{P}} = \underline{\underline{P}}^T$ symmetric

$\underline{\underline{P}}^2 = \underline{\underline{P}}$

$\underline{\underline{P}}^{\parallel} + \underline{\underline{P}}^{\perp} = \underline{\underline{I}}$

$\underline{\underline{P}}^{\parallel} \underline{\underline{P}}^{\perp} = \underline{\underline{0}}$

Reflections



incoming: $\underline{r} = \underline{r}_n^{\parallel} + \underline{r}_n^{\perp}$

reflected: $\underline{r}' = \underline{r}_n^{\parallel} - \underline{r}_n^{\perp}$

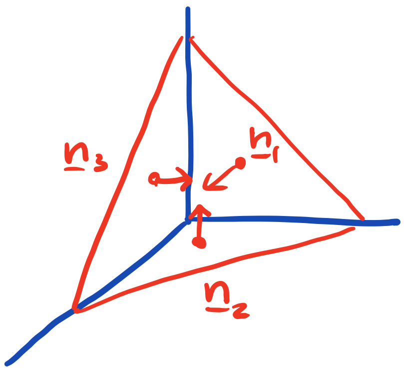
$$\underline{r}' = (\underline{\underline{P}}_n^{\parallel} - \underline{\underline{P}}^{\perp})\, \underline{r}$$

$$\underline{r}' = (\underline{\underline{I}} - 2\, \underline{\hat{n}} \otimes \underline{\hat{n}})\, \underline{r}$$

$$\underline{r}' = \underline{\underline{R}}_n\, \underline{r}$$

Reflection tensor: $\boxed{\underline{\underline{R}}_n = \underline{\underline{I}} - 2\, \underline{\hat{n}} \otimes \underline{\hat{n}}}$

Example: Corner reflector



Inverts the direction of any ray that reflects off all three surfaces.

Direction of triply reflected ray:

$$\underline{r}''' = \underline{\underline{R}}_{n_1} \underline{\underline{R}}_{n_2} \underline{\underline{R}}_{n_3} \underline{r}$$

Show that $\underline{r}''' = -\underline{r}$!