Second-order tensors

Here we are interested in second-order tensors

Linear operators : $\underline{v} = \underline{\underline{A}}\,\underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

Linearity requires that

1) $\underline{\underline{A}}(\underline{u}+\underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$ for all $\underline{u}, \underline{v} \in \mathcal{V}$

2) $\underline{\underline{A}}(\alpha\,\underline{v}) = \alpha\,\underline{\underline{A}}\,\underline{v}$ for all $\alpha \in \mathbb{R}$ and $\underline{v} \in \mathcal{V}$

Example: $\underline{\underline{A}}$ maps every $\underline{v} \in \mathcal{V}$ into $\underline{n} \neq \underline{0} \in \mathcal{V}$.

Is $\underline{\underline{A}}$ a tensor?

Consider $\underline{u}, \underline{v}, \underline{w} \in \mathcal{V}$

$\underline{w} = \underline{u} + \underline{v}$

$$\underline{\underline{A}}(\underline{u}+\underline{v}) \stackrel{?}{=} \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{\underline{A}}\,\underline{w} \stackrel{?}{=} \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{n} \neq \underline{n} + \underline{n}$$

$\Rightarrow$ $\underline{\underline{A}}$ is <u>not</u> a tensor, because it is not linear

Tensor algebra

For all $\underline{v} \in \mathcal{V}$ we define

1) $(\alpha \underline{\underline{A}})\underline{v} = \underline{\underline{A}}(\alpha \underline{v})$ scalar multiplication
2) $(\underline{\underline{A}} + \underline{\underline{B}})\underline{v} = \underline{\underline{A}}\underline{v} + \underline{\underline{B}}\underline{v}$ tensor sum
3) $(\underline{\underline{A}}\,\underline{\underline{B}})\underline{v} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{v})$ tensor product

Note there is also a scalar product introduced later.

The set of all 2^{nd}-order tensors $\mathcal{V}^2$ is a vector space

1) $\alpha \underline{\underline{A}} \in \mathcal{V}^2$ for all $\underline{\underline{A}} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$
2) $\underline{\underline{A}} + \underline{\underline{B}} \in \mathcal{V}^2$ for all $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$
3) $\underline{\underline{A}}\,\underline{\underline{B}} \in \mathcal{V}^2$ for all $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$

Any of these operations will produce another second-order tensor.

Q: What is a basis for $\mathcal{V}^2$?

Two tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are equal if

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \quad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Representation of a tensor

In a frame $\{\underline{e}_i\}$ a second order tensor $\underline{\underline{S}}$ is represented by nine numbers

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Matrix representation of tensor in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Note that $[\underline{\underline{S}}]_{ij} = S_{ij}$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ where $\underline{v} = v_k \underline{e}_k$, $\underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}}(u_j \underline{e}_j) = \underline{\underline{S}}\,\underline{e}_j\, u_j$$

multiply by $\underline{e}_i$ from left

$$v_k\, \underline{e}_i \cdot \underline{e}_k = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_k\, \delta_{ik} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_i = (\underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j)\, u_j \qquad \boxed{v_i = S_{ij}\, u_j}$$

Dyadic Product

The dyadic product of two vectors $\underline{a}$ and $\underline{b}$ is the 2^{nd}-order tensor $\underline{a} \otimes \underline{b}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\,\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

This has the form: $\underline{\underline{A}}\,\underline{v} = \alpha\, \underline{a}$

in components: $A_{ij} v_j = \alpha\, a_i$

$$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$$

$$A_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij}\, v_j = b_j v_j\, a_i$$

$$[\underline{a} \otimes \underline{b}]_{ij}\, v_j = (a_i\, b_j)\, v_j$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$$

So that

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\, \underline{b}^T}$$

Linearity of dyadic product:

for scalars $\alpha, \beta \in \mathbb{R}$ and vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w} \in \mathcal{V}$

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha\, \underline{v} + \beta\, \underline{w}) = \alpha (\underline{a} \otimes \underline{b}) \underline{v} + \beta (\underline{a} \otimes \underline{b}) \underline{w}}$$

The product of two dyadic products

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}} \Rightarrow \text{HW2}$$

needed for tensor product.

Basis for $\mathcal{V}^2$

Given any frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for $\mathcal{V}^2$.

Any second-order tensor $\underline{\underline{S}}$ can be written as linear combination

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j}$$

where $S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ with $\underline{v} = v_i\, \underline{e}_i$, $\underline{u} = u_k \underline{e}_k$

$$v_i \underline{e}_i = S_{ij} (\underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k)$$

$$= S_{ij}\, u_k\, (\underline{e}_i \otimes \underline{e}_j) \cdot \underline{e}_k \quad \text{apply def. of dyadic}$$

$$= S_{ij}\, u_k\, (\underline{e}_j \cdot \underline{e}_k)\, \underline{e}_i = S_{ij}\, u_k\, \delta_{kj}\, \underline{e}_i$$

$$v_i\, \underline{e}_i = S_{ij}\, u_j\, \underline{e}_i$$

Index notation for tensor-vector multiplication

$$\boxed{v_i = S_{ij}\, u_j} \quad \text{used often}$$

Note: Transfer property of kroneckerdelta

$$v_i = \delta_{ij} u_j = u_i$$

also applies to indices of tensors

for example above

$$S_{ij} u_k \delta_{kj} \underline{e}_i =$$

$$u_k \underbrace{S_{ij} \delta_{kj}}_{S_{ik}} \underline{e}_i = S_{ik} u_k \underline{e}_i$$