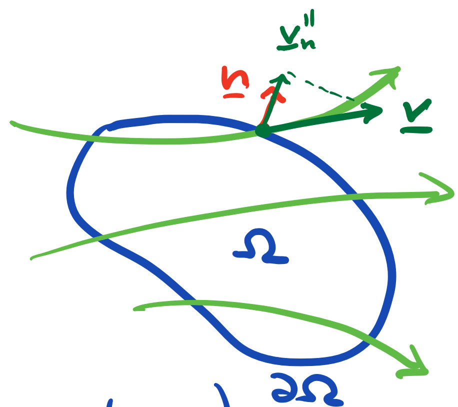
Integral theorems

Essential to derive balance laws

Vector divergence theorem

For any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

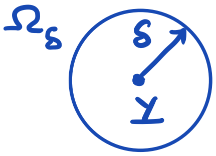
$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{v} \cdot \underline{n} \, dA &= \int_{\Omega} \nabla \cdot \underline{v} \, dV \\ \int_{\partial\Omega} v_i n_i \, dA &= \int_{\Omega} v_{i,i} \, dV \end{aligned}}$$



(for proof see vector calculus class)

Physical Interpretation:

Here $\underline{v}$ is either a velocity $\left[\frac{L}{T}\right]$ or a volumetric flux $\left[\frac{L^3}{L^2 T} = \frac{L}{T}\right]$. The units of $\int_{\partial\Omega} \underline{v} \cdot \underline{n} \, d\Omega$ are then $\left[\frac{L^3}{T}\right]$ so that the l.h.s. represents the rate at which volume is leaving or entering Ω.

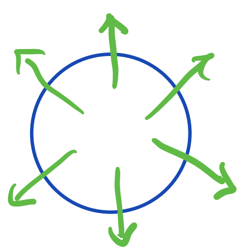


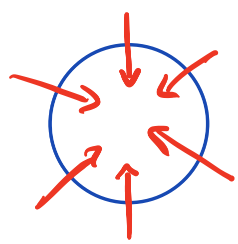
$$\int_{\partial\Omega_\delta} \underline{v}\cdot\underline{n}\, dA = \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV$$

$$\lim_{\delta\to 0}\int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV = V_\delta\, \nabla\cdot\underline{v}\big|_{\underline{x}}$$ V_δ = vol. of sphere

$$\boxed{\nabla\cdot\underline{v}\big|_{\underline{x}} = \lim_{\delta\to 0}\frac{1}{V_\delta}\int_{\partial\Omega}\underline{v}\cdot\underline{n}\, dA}$$

Divergence is the point wise rate of volume expansion/contraction.

$\nabla\cdot\underline{v} > 0$ 

 $\nabla\cdot\underline{v} < 0$

Incompressible flows/deformations are solenoidal $\nabla\cdot\underline{v} = 0$.

Tensor divergence theorem

For any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ on domain Ω with boundary $\partial\Omega$ we have

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA &= \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV \\ \int_{\partial\Omega} S_{ij} n_j \, dA &= \int_{\Omega} S_{ij,j}\, dV \end{aligned}}$$

To derive this from vector divergence Thm consider arbitrary constant vector $\underline{a} \in \mathcal{V}$

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA = \int_{\partial\Omega} \underline{a} \cdot \underline{\underline{S}}\, n \, dA = \int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \underline{n}\, dA$$

where $\underline{\underline{S}}^T \underline{a}$ is a vector and we can apply vector divergence Thm

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} \nabla \cdot (\underline{\underline{S}}^T \underline{a})\, dV$$

using the definition: $(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})$

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} (\nabla \cdot \underline{\underline{S}}) \cdot \underline{a}\, dV$$

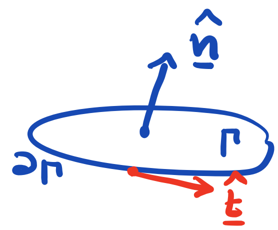
using def. of transpose and that $\underline{a}$ is const.

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\hat{\underline{n}}\, dA = \underline{a} \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV$$

the result follows from arbitrariness of $\underline{a}$

Stokes Thm

Consider surface Γ with boundary $\partial\Gamma$, unit normal $\hat{\underline{n}}$ and unit tangent (right-handed).

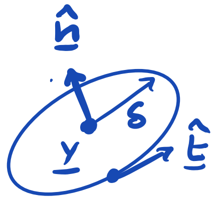


Then for any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

$$\boxed{\int_{\Gamma} (\nabla \times \underline{v}) \cdot \hat{\underline{n}}\, dA = \oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds}$$

Here $\oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds$ is the circulation of $\underline{v}$ around $\partial\Gamma$.

Physical Interpretation:



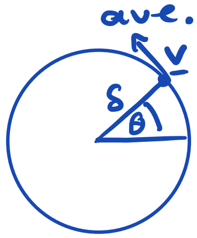
Γ_δ is a disk of radius δ around $\underline{y}$.

$$\oint_{\partial\Gamma} \underline{v}(\underline{x})\cdot\hat{\underline{t}}(\underline{x})\,ds = \int_{\Gamma}(\nabla\times\underline{v})(\underline{x})\cdot\hat{\underline{n}}\,dA$$

In the limit of $\delta\to 0$

$$\underbrace{\overline{\underline{v}\cdot\hat{\underline{t}}}\Big|_{\underline{y}}}\,2\pi\delta \approx \nabla\times\underline{v}\Big|_{\underline{y}}\cdot\hat{\underline{n}}\;\pi\delta^2$$

ave. tangential velocity ~ angular velocity



angular velocity: $\omega = \dfrac{d\theta}{dt}$

$$|\underline{v}| = \omega\delta$$

$$\Rightarrow\quad \overline{\underline{v}\cdot\hat{\underline{t}}}\Big|_{\underline{y}} = \omega\delta$$

$$2\pi\delta^2\omega \approx \nabla\times\underline{v}\Big|_{\underline{y}}\cdot\hat{\underline{n}}\,\pi\delta^2$$

$$2\omega = \nabla\times\underline{v}\Big|_{\underline{y}}\cdot\hat{\underline{n}} \qquad \hat{\underline{n}} = \frac{\nabla\times\underline{v}}{|\nabla\times\underline{v}|}\Bigg|_{\underline{y}}$$

$$2\omega = \frac{(\nabla\times\underline{v}|_{\underline{y}})\cdot(\nabla\times\underline{v}|_{\underline{y}})}{|\nabla\times\underline{v}|_{\underline{y}}} = |\nabla\times\underline{v}|_{\underline{y}}$$

$$\Rightarrow\quad \boxed{|\nabla\times\underline{v}|_{\underline{y}} = 2\omega}$$

Curl of $\underline{v}$ is twice the angular velocity.

Derivatives of tensor functions

So far we have considered fields: $\phi(\underline{x}), \underline{v}(\underline{x}), \underline{\underline{S}}(\underline{x})$

Now we are interested in tensor functions

- scalar-valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor-valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar-valued tensor functions

Typical examples: $\det \underline{\underline{A}}$ or $\operatorname{tr} \underline{\underline{A}}$

Definition: A function $\psi(\underline{\underline{S}})$ is differentiable at $\underline{\underline{A}}$ if there exists a tensor $D\psi(\underline{\underline{A}})$, s.t.

$$\psi(\underline{\underline{A}} + \underline{\underline{H}}) = \psi(\underline{\underline{A}}) + D\psi(\underline{\underline{A}}) : \underline{\underline{H}} + o(|\underline{\underline{H}}|)$$

or equivalently with $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$

$$\boxed{D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{\underline{U}} \in \mathcal{V}^2$$

$D\psi(\underline{\underline{A}})$ is called the derivative of ψ at $\underline{\underline{A}}$

In frame $\{\underline{e}_i\}$ we have $\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j}$

To see this write $\psi(A_{11}, A_{12}, \dots, A_{33})$

and $\underline{\underline{U}} = U_{kl}\, \underline{e}_k \otimes \underline{e}_l$ then

$$\psi(\bar{\underline{\underline{A}}} + \epsilon \underline{\underline{U}}) = \psi(\underbrace{\bar{A}_{11} + \epsilon U_{11}}_{A_{11}}, \underbrace{\bar{A}_{12} + \epsilon U_{12}}_{A_{12}}, \dots, \bar{A}_{33} + \epsilon U_{33})$$

by chain rule

$$D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\bar{A}_{11} + \epsilon U_{11}, \dots, \bar{A}_{33} + \epsilon U_{33}) \Big|_{\epsilon = 0}$$

$$= \frac{\partial \psi}{\partial A_{11}} U_{11} + \frac{\partial \psi}{\partial A_{12}} U_{12} + \dots + \frac{\partial \psi}{\partial A_{33}} U_{33} = \frac{\partial \psi}{\partial A_{ij}} U_{ij}$$

$$= \left(\frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j \right) : \left(U_{kl}\, \underline{e}_k \otimes \underline{e}_l \right)$$

result is implied by the arbitraryness of $\underline{\underline{U}}$

Derivative of trace

$\psi(\underline{\underline{A}}) = \mathrm{tr}(\underline{\underline{A}}) = A_{ii}$ Using the definition

$$D\mathrm{tr}(\underline{\underline{A}}) = \frac{\partial A_{ii}}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \delta_{ik} \delta_{il} \underline{e}_k \otimes \underline{e}_l = \underline{e}_i \otimes \underline{e}_i = \underline{\underline{I}}$$

$$\boxed{D\mathrm{tr}(\underline{\underline{A}}) = \underline{\underline{I}}}$$

Derivative of determinant

Let $\psi(\underline{\underline{A}}) = \det(\underline{\underline{A}})$, if $\underline{\underline{A}}$ is invertible

$$\boxed{D\det(\underline{\underline{A}}) = \det(\underline{\underline{A}})\, \underline{\underline{A}}^{-T}}$$

Note this takes some work !

Start by using the directional derivative

$$D\det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) = \frac{d}{d\epsilon} \det(\underline{\underline{A}} + \epsilon\underline{\underline{U}})\Big|_{\epsilon=0}$$

First simplify expansion

$$\det(\epsilon\underline{\underline{U}} + \underline{\underline{A}}) = \det\left(\epsilon\underline{\underline{A}}(\underline{\underline{A}}^{-1}\underline{\underline{U}} + \tfrac{1}{\epsilon}\underline{\underline{I}})\right) \qquad \frac{1}{\epsilon} = -\lambda$$

$$= \det(\epsilon\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

$$= \epsilon^3 \det(\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

from definition of principal invariants

$$\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}}) = -\lambda^3 + \lambda^2 I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) - \lambda I_2(\underline{\underline{A}}^{-1}\underline{\underline{B}}) + I_3(\underline{\underline{A}}^{-1}\underline{\underline{B}})$$

$$= -\left(-\frac{1}{\epsilon}\right)^3 + \left(-\frac{1}{\epsilon}\right)^2 I_1 + \frac{1}{\epsilon} I_2 + I_3$$

$$= \frac{1}{\epsilon^3} + \frac{1}{\epsilon^2} I_1 + \frac{1}{\epsilon} I_2 + I_3$$

substituting above $\Rightarrow$ expansion in ϵ

$$\det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) = \epsilon^3 \det(\underline{\underline{A}}) \left(\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2} I_1 + \frac{1}{\epsilon} I_2 + I_3 \right)$$

$$= \det(\underline{\underline{A}}) \left(1 + \epsilon I_1 + \epsilon^2 I_2 + \epsilon^3 I_3 \right)$$

substitute into directional derivative

$$D\det(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon = 0} = \det(\underline{\underline{A}}) I_1(\underline{\underline{A}}^{-1} \underline{\underline{U}})$$

since $I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(A^{-1}_{ij} U_{jk}\, \underline{e}_i \otimes \underline{e}_k)$

$$= A^{-1}_{ij} U_{ji} = A^{-T}_{ji} U_{ji} = \underline{\underline{A}}^{-T} : \underline{\underline{U}}$$

so that

$$D\det(\underline{\underline{A}}) : \underline{\underline{U}} = \det(\underline{\underline{A}})\, \underline{\underline{A}}^{-T} : \underline{\underline{U}}$$

the result follows from the arbitraryness of $\underline{\underline{U}}$

Time derivative of scalar valued tensor function

Let $\underline{\underline{S}} = \underline{\underline{S}}(t) \in \mathcal{V}^2$. In stationary frame $\{\underline{e}_i\}$

$$\underline{\underline{S}}(t) = S_{ij}(t)\, \underline{e}_i \otimes \underline{e}_j \quad \text{so that}$$

$$\boxed{\dot{\underline{\underline{S}}} = \frac{d\underline{\underline{S}}}{dt} = \frac{dS_{ij}}{dt}\, \underline{e}_i \otimes \underline{e}_j}$$

How do we compute $\frac{d}{dt}\psi(\underline{\underline{S}}(t))$?

By the chain rule we have

$$\begin{aligned}\frac{d}{dt}\psi(\underline{\underline{S}}(t)) &= \frac{d}{dt}\psi\big(S_{11}(t), S_{12}(t), \dots, S_{33}(t)\big)\\ &= \frac{\partial\psi}{\partial S_{11}}\frac{dS_{11}}{dt} + \dots + \frac{\partial\psi}{\partial S_{33}}\frac{dS_{33}}{dt} = \frac{\partial\psi}{\partial S_{ij}}\frac{dS_{ij}}{dt}\\ &= D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}\end{aligned}$$

$\Rightarrow$ chain rule leads to a contraction

$$\boxed{\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}}$$

Example: $\boxed{\frac{d}{dt}\det(\underline{\underline{S}}(t)) = \det(\underline{\underline{S}})\, \underline{\underline{S}}^{-T} : \dot{\underline{\underline{S}}}}$ Jacobi's formula