Polar decomposition

Any tensor $\underline{\underline{F}} \in \mathcal{V}^2$ with $\det(\underline{\underline{F}}) > 0$ has a right and left polar decomposition

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$$

$\underline{\underline{R}}$ = rotation

$$\left.\begin{array}{l} \underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}} \\ \underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} \end{array}\right\}$$

symmetric positive definite

$\underline{v} \cdot \underline{\underline{S}}\,\underline{v} > 0$ for all $\underline{v} \in \mathcal{V}$

$\Rightarrow \lambda_i > 0 \qquad \lambda_i \in \mathbb{R}$

$$\underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} \quad \Rightarrow \quad \boxed{\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}}$$

Show $\underline{\underline{R}}$ is rotation:

1) $\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}$ (orthonormal)

2) $\det(\underline{\underline{R}}) = 1$ (rotation)

Note: $(\underline{\underline{A}}^T)^{-1} = (A^{-1})^T \quad \Rightarrow \quad \underline{\underline{A}} = \underline{\underline{A}}^T$

symmetric: $\underline{\underline{A}} = \underline{\underline{A}}^T \quad \Rightarrow \quad \underline{\underline{A}}^{-1} = (\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^T$

1) Orthonormal

$$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-T}\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{\underline{\underline{U}}^2}\underline{\underline{U}}^{-1} = \underbrace{\underline{\underline{U}}^{-1}\underline{\underline{U}}}_{\underline{\underline{I}}}\,\underbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}_{\underline{\underline{I}}} = \underline{\underline{I}}$$

2) Rotation

$$\det(R) = \det(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \frac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$$

$\det(\underline{\underline{F}}) > 0$ for admissible deformation

$\det(\underline{\underline{U}}) = \lambda_1\lambda_2\lambda_3 > 0$ if $\underline{\underline{U}}$ is s.p.d.

<u>Show $\underline{\underline{U}}$ is symmetric positive definite</u>

$|\underline{a}|^2 = \underline{a}\cdot\underline{a} > 0$ $\qquad\qquad$ $\underline{a} = \underline{\underline{F}}\,\underline{v}$

$$(\underline{\underline{F}}\,\underline{v})\cdot(\underbrace{\underline{\underline{F}}\,\underline{v}}_{\underline{a}}) > 0$$

$$\underline{\underline{F}}\,\underline{v}\cdot\underline{a} = \underline{v}\cdot\underline{\underline{F}}^T\underline{a} = \underline{v}\cdot\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{v} = \underline{v}\cdot\underline{\underline{U}}^2\underline{v} = \underline{v}\cdot\underline{\underline{C}}\,\underline{v} > 0$$

$\Rightarrow$ $\underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{U}}^2 = \underline{\underline{C}}$ is s.p.d.

eigenvalues of $\underline{\underline{C}}$ $\lambda_i > 0$

Is $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$ also s.p.d. ?

<u>Tensor square root</u>

If $\underline{\underline{C}}$ is a s.p.d. tensor with eigenpair $(\lambda, \underline{v})$

$$\boxed{\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\ \underline{v}_i \otimes \underline{v}_i}$$

eigenpair of $\underline{\underline{U}}$ is $(\omega, \underline{v})$ where $\omega_i = \sqrt{\lambda_i} > 0$

$\Rightarrow\ \underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is s.p.d.

Similarly $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ is s.p.d.

Analysis of local deformation

Any $\underline{\varphi}(\underline{X})$ can be approximated locally as a homogeneous affine deformation.

$$\boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}} \quad \text{where} \quad \underline{\underline{F}} = \nabla \underline{\varphi}$$

$\underline{\underline{F}}$ is a measure of strain but it is not suitable as strain tensor, because it contains rotations that do not lead to deformation.

Building strain tensor is 3 step process

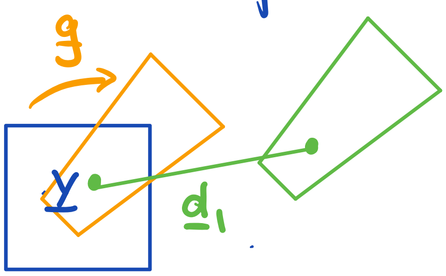
1) Remove translations
2) Remove rotations
3) Find principal streches

1) Translation - fixed point decomposition

Any hom. φ can be decomposed as

$$\boxed{\varphi = \underline{d}_1 \circ g = g \circ \underline{d}_2}$$

where $g = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ is a hom. def with fixed point $\underline{Y}$ and $\underline{d}_i = \underline{X} + \underline{a}_i$ with $i = \{1, 2\}$ are translations from $\underline{Y}$.



$$\underline{y} = g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\underline{y} = g(\underline{Y}) = \underline{Y}$$

Rewrite φ as "Taylor expansion" around $\underline{Y}$

Consider points $\underline{X}$ and $\underline{Y}$ and their maps

$$\underline{x} = \underline{c} + \underline{\underline{F}}\,\underline{X} \qquad \text{and} \qquad \underline{y} = \underline{c} + \underline{\underline{F}}\,\underline{Y}$$

subtracting $\underline{x} - \underline{y} = \underline{\underline{F}}(\underline{X} - \underline{Y})$ or

$$\boxed{\varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})} \Leftrightarrow \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}$$

Like a Taylor series but for hom. def. this is true even if $|\underline{X} - \underline{Y}|$ is not small.

Given $\underline{g}(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ and

$\underline{d}_i(\underline{X}) = \underline{X} + \underline{a}_i \qquad i = 1, 2$

$(\underline{d}_1 \circ \underline{g})(\underline{X}) = \underline{d}_1(\underline{g}(\underline{X})) = \underline{g}(\underline{X}) + \underline{a}_1$

$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{a}_1$

choose $\underline{a}_1 = \underline{\varphi}(\underline{Y}) - \underline{Y}$, translation of fixedpoint.

note φ itself does not have a fixedpoint!

substitute

$(\underline{d}_1 \circ \underline{g})(\underline{X}) = \cancel{\underline{Y}} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{\varphi}(\underline{Y}) - \cancel{\underline{Y}}$

$= \underline{\varphi}(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y}) = \underline{\varphi}(\underline{X})$ (from box above)

$\Rightarrow \quad \underline{\varphi}(\underline{X}) = (\underline{d}_1 \circ \underline{g})(\underline{X})$ ✓

$\Rightarrow$ always extract translation and assume that our def. has a fixed point.

Strech-rotation decomposition

Let $\varphi(\underline{X})$ be hom. def. with fixed point $\underline{Y}$ so that $\varphi(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ then we have

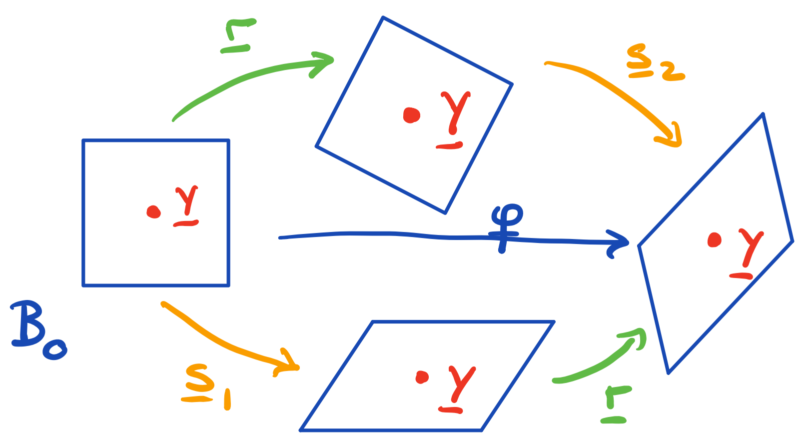
$$\boxed{\underline{\varphi} = \underline{r} \circ \underline{s}_1 = \underline{s}_2 \circ \underline{r}}$$

where $\underline{r} = \underline{Y} + \underline{\underline{R}}(\underline{X} - \underline{Y})$ is a rotation around $\underline{Y}$

$$\left.\begin{aligned} \underline{s}_1 &= \underline{Y} + \underline{\underline{U}}(\underline{X} - \underline{Y}) \\ \underline{s}_2 &= \underline{Y} + \underline{\underline{V}}(\underline{X} - \underline{Y}) \end{aligned}\right\} \text{ streches from } \underline{Y}$$

The tensors $\underline{\underline{R}}$, $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are given by polar decomposition

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$



To see this consider

$$(\underline{r} \circ \underline{s}_1)(\underline{X}) = \underline{r}(\underline{s}_1(\underline{X})) = \underline{Y} + \underline{\underline{R}}(\underline{s}_1(\underline{X}) - \underline{Y})$$

$$= \underline{Y} + \underline{\underline{R}}(\cancel{\underline{Y}} + \underline{\underline{U}}(\underline{X} - \underline{Y}) - \cancel{\underline{Y}})$$

$$= \underline{Y} + \underline{\underline{R}}\,\underline{\underline{U}}(\underline{X} - \underline{Y}) =$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$(\underline{r} \circ \underline{s}_1)(\underline{X}) = \underline{\varphi}(\underline{X}) \quad \checkmark$$

for $s_2 \circ r = \varphi$ see PS!

Strech tensors

Both $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

$\Rightarrow$ spectral decomposition

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i \quad \text{and} \quad \underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$

where $\{\lambda_i, \underline{u}_i\}$ and $\{\lambda_i, \underline{v}_i\}$ are eigenpairs of $\underline{\underline{U}}$ & $\underline{\underline{V}}$

same eigenvalues but different eigenvectors.

Note: $\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}} \;\rightarrow\; \underline{\underline{R}}^T \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}} \;\rightarrow\; \underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}}$

Consider char. polynomial

$$p_u(\lambda) = \det(\underline{\underline{U}} - \lambda \underline{\underline{I}}) = \det(\underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}} - \lambda \underline{\underline{R}}^T \underline{\underline{R}})$$

$$= \det(\underline{\underline{R}}^T (\underline{\underline{V}} - \lambda \underline{\underline{I}}) \underline{\underline{R}}) = \cancel{\det(\underline{\underline{R}}^T)}^{1} \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) \cancel{\det(\underline{\underline{R}})}^{1}$$

$$= \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) = p_v(\lambda)$$

$\Rightarrow$ $\underline{\underline{U}}$ and $\underline{\underline{V}}$ have same eigenvalues

λ_i's are principal streches

$\underline{u}_i$ and $\underline{v}_i$ are right and left principal dir.

The λ_i's give the streching of the body in the $\underline{u}_i$ and $\underline{v}_i$ directions.

What is the relation between $\underline{u}_i$ and $\underline{v}_i$?

$$\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{u}_i$$

$$\underline{\underline{R}}\,\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{\underline{R}}\,\underline{u}_i \qquad F = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$

$$\underline{\underline{V}} \underbrace{\underline{\underline{R}}\,\underline{u}_i}_{\underline{v}_i} = \lambda_i \underbrace{\underline{\underline{R}}\,\underline{u}_i}_{\underline{v}_i}$$

$\boxed{\underline{v}_i = \underline{\underline{R}}\,\underline{u}_i}$ differ by rotation.

In summary:

Any hom. def. φ can be decomposed into a sequence of 3 elementary deformations:

1) Translation
2) Rotation
3) Strech along principal directions

Example: $\varphi = \underline{s}_2 \circ \underline{r} \circ \underline{d}_2$

$\varphi = \underline{r} \circ \underline{s}_1 \circ \underline{d}_2$

...

Note: These results for hom. def. hold for any def. in a small neighborhood by Taylor expansion.

Cauchy-Green Strain Tensor

Consider a deformation $\varphi : \mathcal{B} \to \mathcal{B}'$ with $\underline{\underline{F}} = \nabla \varphi$, then the (right) Cauchy-Green strain tensor is

$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}} \, .$$

Note that $\underline{\underline{C}}$ is always symmetric pos. definite.

The deformation gradient $\underline{\underline{F}}$ contains information about both rotations and streches. Using the right polar decomposition we have

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$$

$\underline{\underline{R}}$ is rotation matrix

$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is right strech tensor

Clearly $\underline{\underline{C}} = \underline{\underline{U}}^2$ and the rotation $\underline{\underline{R}}$ implicit in $\underline{\underline{F}}$ is not present in in $\underline{\underline{C}}$.

$\Rightarrow$ $\boxed{\text{The right Cauchy Green strain tensor only contains information about streches.}}$

Hence we can cannot obtain $\underline{\underline{F}}$ from $\underline{\underline{C}}$!

Remarks:

1) Strictly the right-stretch tensor $\underline{\underline{U}}$ is sufficient. We introduce $\underline{\underline{C}} = \underline{\underline{U}}^2$ to avoid the tensor square root.

Simple example:

$$[\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$[\underline{\underline{C}}] = [\underline{\underline{F}}^T][\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To get $[\underline{\underline{U}}]$ we need to solve eigenvalue problem

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 5-\mu & 4 \\ 0 & 4 & 5-\mu \end{vmatrix} = (1-\mu)(5-\mu)^2 - 16(1-\mu) = 0$$

Eigenvalues: $\mu_{1,2} = 1 \quad \mu_3 = 9$

Eigenvectors: $[\underline{u}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [\underline{u}_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \underline{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Hence: $[\underline{\underline{U}}] = \sqrt{[\underline{\underline{C}}]} = \sum_{i=1}^{3} \sqrt{\mu_i}\, \underline{u}_i \otimes \underline{u}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2) $\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i$ where

λ_i's are principal stretches

$\underline{u}_i$'s are right principal directions

$\underline{\underline{C}} = \underline{\underline{U}}^2 = \sum_{i=1}^{3} \lambda_i^2 \, \underline{u}_i \otimes \underline{u}_i$

$\mu_i = \lambda_i^2$ eig. values of $\underline{\underline{C}}$ are squares of principal streches

eigenvectors are right principal dir.

3) $C_{KL} = F_{iK} F_{iL}$ "material strain tensor"

spatial indices are contracted

Other strain tensors

I) $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$: Green-Lagrange tensor

$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$ material tensor $\Rightarrow$ linear theory

II) $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{V}}^2$: Left Cauchy-Green tensor

$b_{kl} = F_{kI} F_{lI}$ "spatial tensor" (a.k.a. Finger tensor)

III) $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1})$: Euler-Almansi tensor

$e_{kl} = \frac{1}{2}(\delta_{kl} - F^{-1}_{Ik} F^{-1}_{Il})$ "spatial tensor"