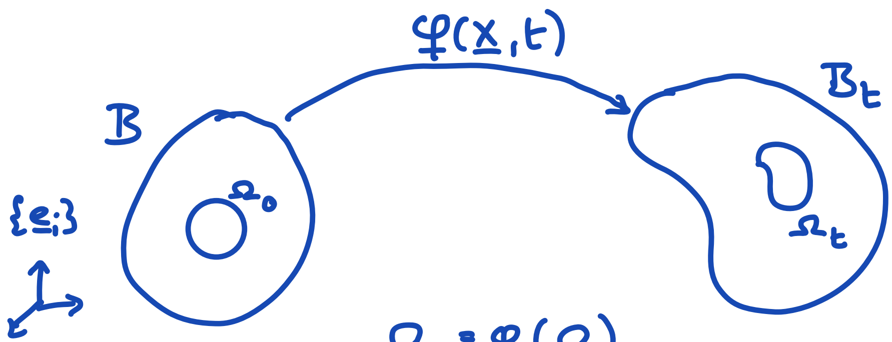
Local Eulerian Balance Laws



$\Omega_t = \underline{\varphi}(\Omega)$

Arbitrary subset Ω

I) Conservation of mass

Mass of Ω_t : $M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x$

no reactions or relativistic effects $\Rightarrow$ mass is constant

$$\boxed{\frac{d}{dt} M[\Omega_t] = 0} \quad \Rightarrow \quad M[\Omega_t] = M[\Omega_0]$$

using $dV_x = J(\underline{X},t)\, dV_X$ where $J = \det(\underline{F})$

$$M[\Omega_0] = M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x = \int_{\Omega_0} \underbrace{\rho(\underline{\varphi}(\underline{X},t),t)}_{\rho_m(\underline{X},t)}\, J(\underline{X},t)\, dV_X$$

$$M[\Omega_0] = \int_{\Omega_0} \rho_m(\underline{X},t)\, J(\underline{X},t)\, dV_X$$

At $t=0$: $\Omega_t \to \Omega_0$ $\quad J(\underline{X},0)=1 \qquad \underline{x}=\underline{X}$

$$M[\Omega_0] = \int_{\Omega_0} \rho_m(\underline{X},0)\,dV_X = \int_{\Omega_0} \rho_0(\underline{X})\,dV_X \qquad \rho_0 = \text{initial mass}$$

$$\Rightarrow \quad M[\Omega_t] = \int_{\Omega_0} \rho_m(\underline{X},t)\,J(\underline{X},t)\,dV_X = \int_{\Omega_0} \rho_0(\underline{X})\,dV_X$$

$$\int_{\Omega_0} \left[\rho_m(\underline{X},t)\,J(\underline{X},t) - \rho_0(\underline{X})\right] dV_X = 0$$

by arbitrariness of Ω we have

$$\boxed{\rho_m(\underline{X},t)\,J(\underline{X},t) = \rho_0(\underline{X})}$$

Lagraugian statement of mass conservation. $(\underline{X})$

Convert to Euleriau: $\frac{\partial}{\partial t}$

$$\underbrace{\frac{\partial}{\partial t}\rho_m(\underline{X},t)}_{\dot\rho(\underline{x},t)}\,J(\underline{X},t) + \rho_m(\underline{X},t)\underbrace{\frac{\partial}{\partial t}J(\underline{X},t)}_{J(\nabla_x\cdot\underline{v})_m} = \cancel{\frac{\partial}{\partial t}\rho_0(\underline{X})}^{\,0} = 0$$

dividing by J and switching to spatial description

$$\dot\rho(\underline{x},t) + \rho(\underline{x},t)\,\nabla_x\cdot\underline{v}(\underline{x},t) = 0$$

$$\Rightarrow \quad \boxed{\dot\rho + \rho\,\nabla_x\cdot\underline{v} = 0} \quad \text{local Euleriau form}$$

expanding the material derivative we have

$$\frac{\partial}{\partial t}\rho + \underbrace{\nabla_x \rho \cdot \underline{v} + \rho\, \nabla_x \cdot \underline{v}}_{\nabla_x \cdot (\rho \underline{v})} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \underline{v}) = 0}$$ <u>conservative</u> local Eulerian form

Time derivative of integrals relative to mass

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega_t} \dot{\phi}(\underline{x},t)\rho(\underline{x},t)\, dV_x}$$

where $\phi(\underline{x},t)$ is any spatial scalar, vector or tensor field.

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega} \phi_m(\underline{X},t)\underbrace{\rho_m(\underline{X},t)\det \underline{\underline{F}}(\underline{X},t)}_{\rho_0(\underline{X})}\, dV_X$$

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x = \int_{\Omega} \phi_m(\underline{X},t)\rho_0(\underline{X})\, dV_X$$

Take derivative

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\, dV_x &= \int_{\Omega} \frac{d}{dt}\phi_m(\underline{X},t)\, \rho_0(X)\, dV_X \\ &= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\, \rho_0(\underline{X})\, dV_X \\ &= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\, \rho_m(\underline{X},t)\, \det \underline{\underline{F}}(\underline{X},t)\, dV_X \\ &= \int_{\Omega_t} \dot{\phi}(\underline{x},t)\, \rho(\underline{x},t)\, dV_x \quad ✓\end{aligned}$$

Laws of inertia (Galileo & Newton)

linear momentum: $\underline{l}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$

angular momentum: $\underline{j}[\Omega_t]_z = \int_{\Omega_t} (\underline{x}-\underline{z}) \times \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$

With respect to a fixed frame of reference, the rate of change of linear and angular momentum of any $\Omega_t \subseteq B_t$ equal the resultant force & torque about the origin

$$\boxed{\begin{aligned} \frac{d}{dt}\underline{l}[\Omega] &= \int_{\Omega_t} \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{t}(\underline{x},t)\,dA_x \\ \frac{d}{dt}\underline{j}[\Omega]_0 &= \int_{\Omega_t} \underline{x} \times \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{x} \times \underline{t}(\underline{x},t)\,dA_x \end{aligned}}$$

II) Balance of Linear momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{t}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

where ρ, $\underline{v}$, $\underline{t}$ and $\underline{b}$ are spatial fields.

Cauchy stress field: $\underline{t} = \underline{\underline{\sigma}}\, \underline{n}$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

using tensor divergence theorem

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\Omega_t} \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}\, dV_x$$

using derivative relative to mass

$$\int_{\Omega_t} \rho \dot{\underline{v}} - \nabla \cdot \underline{\underline{\sigma}} - \rho \underline{b}\, dV_x = 0$$

by the arbitraryness of Ω_t, we have

$$\boxed{\rho \dot{\underline{v}} - \nabla \cdot \underline{\underline{\sigma}} = \rho \underline{b}}$$ local Eulerian form

Also referred to as <u>Cauchy's first equation of motion</u>.

To rewrite this in conservative form consider the following

$$\rho \dot{\underline{v}} = \rho \frac{\partial \underline{v}}{\partial t} + \rho (\nabla_x \underline{v}) \underline{v} = \frac{\partial}{\partial t}(\rho \underline{v}) - \frac{\partial \rho}{\partial t} \underline{v} + (\nabla_x \underline{v})(\rho \underline{v})$$

using mass balance $\frac{\partial \rho}{\partial t} = -\nabla_x \cdot (\rho \underline{v})$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v}) \underline{v} + (\nabla_x \underline{v})(\rho \underline{v})$$

using $\nabla \cdot (\underline{a} \otimes \underline{b}) = (\nabla \underline{a}) \underline{b} + \underline{a} \, \nabla \cdot \underline{b}$ (see HW5 Q4)

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v})$$

Hence we have conservative local Eulerian form

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

conserved quantity: $\rho \underline{v}$ = linear momentum

advective mom. flux: $\rho \underline{v} \otimes \underline{v}$

diffusive mom. flux: $-\underline{\underline{\sigma}}$

III) Balance of angular momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{t}\, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b}\, dV_x$$

The left hand side becomes

$$\frac{d}{dt}\int_{\Omega_t} \rho(\underline{x} \times \underline{v})\, dV_x = \int_{\Omega_t} \rho \frac{d}{dt}(\underline{x} \times \underline{v})\, dV_x =$$

$$= \int_{\Omega_t} \rho(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}})\, dV_x \qquad \dot{\underline{x}} = \underline{v}$$

$$\underline{v} \times \underline{v} = \underline{0}$$

$$= \int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x$$

Substituting Cauchy stress field the r.h.s becomes

$$\int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho(\underline{x} \times \underline{b})\, dV_x$$

$$\int_{\Omega_t} \underline{x} \times (\rho\dot{\underline{v}} - \rho \underline{b})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x$$

substitute linear mom. balance $\rho\dot{\underline{v}} - \rho\underline{b} = \nabla_x \cdot \underline{\underline{\sigma}}$

$$\boxed{\int_{\Omega_t} \underline{x} \times \nabla_x \cdot \underline{\underline{\sigma}}\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x}$$

This is exactly the statement we had for the static case

in Lecture 14 on Mechanical Equilibrium.

$\Rightarrow$ $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ extends to transient cases.