# Cauchy-Green Strain Tensor

For $\underline{x} = \underline{\varphi}(\underline{X})$ with $\underline{\underline{F}} = \nabla \varphi$ ,

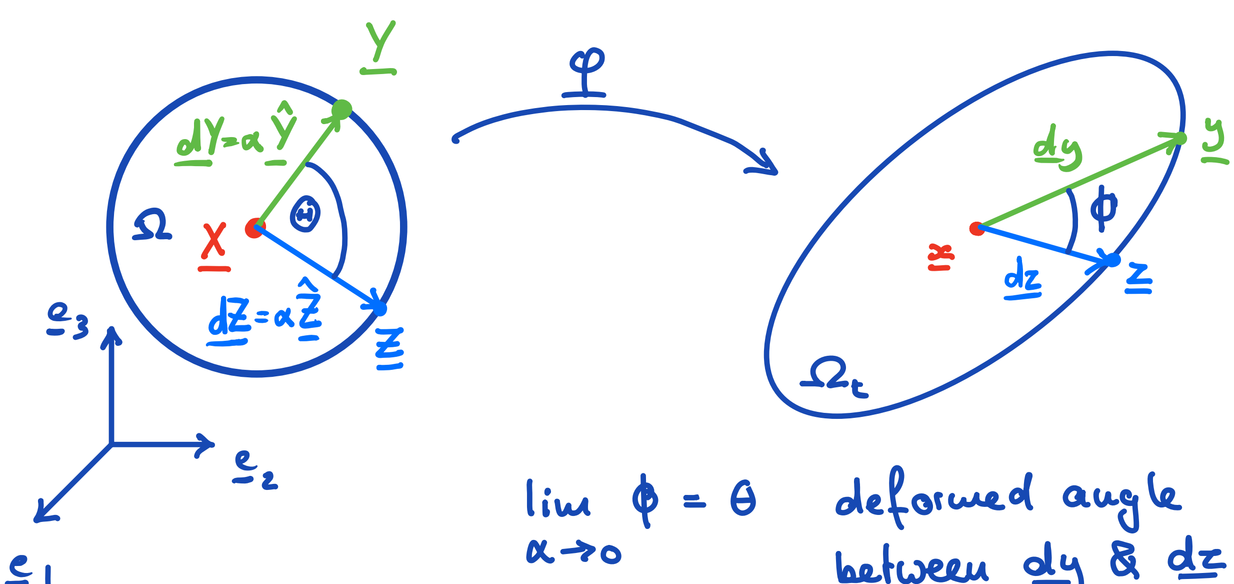
$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2} \qquad \text{sym. pos. def.}$$

$\underline{\underline{U}}$ is right-strech tensor $\qquad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

$\Rightarrow$ only information about streches

# Interpretation of $\underline{\underline{C}}$

How are changes in relative position and orientation of material points quantified by $\underline{\underline{C}}$ ?




$$\lim_{\alpha \to 0} \phi = \theta \qquad \text{deformed angle between } \underline{dy} \text{ \& } \underline{dz}$$

# Cauchy-Green strain relations

For any point $\underline{X} \in B$ and unit vectors $\hat{\underline{Y}}$ and $\hat{\underline{Z}}$ we define $\lambda(\hat{\underline{Y}}) > 0$ and $\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) \in [0, \pi]$ by

$$\boxed{\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Y}}}} \quad \text{and} \quad \boxed{\cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Y}}}\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\hat{\underline{Z}}}}}$$

## I. Streches

In the limit as $\alpha \to 0$ we have

$$\frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} = \frac{|d\underline{y}|}{|d\underline{Y}|} \to \lambda(\hat{\underline{Y}}) \quad \text{and} \quad \frac{|\underline{z} - \underline{x}|}{|\underline{Z} - \underline{X}|} = \frac{|d\underline{z}|}{|d\underline{Z}|} \to \lambda(\hat{\underline{Z}})$$

Therefore $\lambda(\hat{\underline{Y}})$ is the stretch in direction $\hat{\underline{Y}}$ at $\underline{X}$.

A stretch is the ratio of deformed to initial length.

To determine the stretch we use $d\underline{y} = \underline{\underline{F}}(\underline{X})d\underline{Y}$.

$$|d\underline{y}|^2 = d\underline{y} \cdot d\underline{y} = \underline{\underline{F}}d\underline{Y} \cdot (\underline{\underline{F}}d\underline{Y}) = d\underline{Y} \cdot \underline{\underline{F}}^T\underline{\underline{F}}d\underline{Y} = d\underline{Y} \cdot \underline{\underline{C}}d\underline{Y}$$
$$= \alpha^2 \hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Y}}$$

$|d\underline{Y}|^2 = \alpha^2$ by definition

So that $\frac{|d\underline{y}|^2}{|d\underline{Y}|^2} = \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}} = \lambda^2(\hat{\underline{Y}})$

taking square root: $\lambda(\underline{e}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}$ ✓

If $\underline{u}_i$ is a right-principal strech, so that

$$(\underline{\underline{C}} - \lambda_i^2 \underline{\underline{I}})\hat{\underline{u}}_i = 0 \quad \text{(no sum)}$$

$$\hat{\underline{u}}_i \cdot \underline{\underline{C}}\,\hat{\underline{\phantom{u}}}_i - \lambda_i^2\, \hat{\underline{u}}_i \cdot \hat{\underline{u}}_i = 0 \qquad \underline{u}_i \cdot \underline{\underline{C}}\,\hat{\underline{u}}_i = \lambda_i^2$$

note: $\hat{\underline{U}}_i$ is the eigenvector of $\underline{\underline{C}}$ it is capitalized because it is a material vector

then $\lambda(\hat{\underline{u}}_i) = \lambda_i$ which justifies referring to $\lambda_i$'s as principal streches.

Arguments similar to determination of principal stresses show that $\lambda(\hat{\underline{Y}})$ has extremum if $\hat{\underline{Y}} = \hat{\underline{u}}_i$.

## II. Shear

Change in angle

$$\boxed{\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})}$$

$\Theta(\underline{d\hat{Y}}, \underline{d\hat{Z}})$ angle between $\underline{d\hat{Y}}$ & $\underline{d\hat{Z}}$ in initial conf.

$\theta(\underline{d\hat{Y}}, \underline{d\hat{Z}})$ angle between $\underline{dy}$ & $\underline{dz}$ in limit $\alpha \to 0$

$$\boxed{\cos\phi \longrightarrow \cos\theta(\underline{\hat{Y}}, \underline{\hat{Z}})}$$

To see this consider $\cos\phi = \dfrac{\underline{dy}\cdot\underline{dz}}{|\underline{dy}|\,|\underline{dz}|}$

where $\underline{dy}\cdot\underline{dz} = (\underline{\underline{F}}\,\underline{dY})\cdot(\underline{\underline{F}}\,\underline{dZ})$

$$= \underline{dY}\cdot\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{dZ} = \underline{dY}\cdot\underline{\underline{C}}\,\underline{dZ}$$

$$= \alpha^2\,\underline{\hat{Y}}\cdot\underline{\underline{C}}\,\underline{\hat{Z}}$$

with $|\underline{dy}| = \alpha\sqrt{\underline{\hat{Y}}\cdot\underline{\underline{C}}\,\underline{\hat{Y}}}$ and $|\underline{dz}| = \alpha\sqrt{\underline{\hat{Z}}\cdot\underline{\underline{C}}\,\underline{\hat{Z}}}$

substituting into $\cos\phi = \dfrac{\underline{dy}\cdot\underline{dz}}{|\underline{dy}|\,|\underline{dz}|}$

$$\cos\phi = \frac{\underline{d\hat{Y}}\cdot\underline{\underline{C}}\,\underline{d\hat{Z}}}{\sqrt{\underline{d\hat{Y}}\cdot\underline{\underline{C}}\,\underline{d\hat{Y}}}\,\sqrt{\underline{d\hat{Z}}\cdot\underline{\underline{C}}\,\underline{d\hat{Z}}}} \xrightarrow[\alpha\to 0]{} \cos\theta(\underline{d\hat{Y}}, \underline{d\hat{Z}})$$

Compute the shear $\gamma(\underline{\hat{Y}}, \underline{\hat{Z}}) = \Theta(\underline{\hat{Y}}, \underline{\hat{Z}}) - \theta(\underline{\hat{Y}}, \underline{\hat{Z}})$

$\Rightarrow$ interpret components of $\underline{\underline{C}}$

## Components of $\underline{\underline{C}}$

Let $C_{IJ}$ be the components of $\underline{\underline{C}}$ in an arbitrary frame $\{\underline{e}_I\}$, then for any point $\underline{X} \in B$ we have that

$$\boxed{\begin{aligned} C_{II} &= \lambda^2(\underline{e}_I) \\ C_{IJ} &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\,\sin\gamma(\underline{e}_I,\underline{e}_J) \end{aligned}} \quad \text{(no sum)}$$

$\Rightarrow$ The diagonal components of $C$ are the squares of the streches in coord. directions. Off diagonal components are related to shears between coordinate directions.

Components of $\underline{\underline{C}}$:

$$\underline{\underline{C}} = C_{IJ}\,\underline{e}_I \otimes \underline{e}_J \quad \Rightarrow \quad C_{IJ} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_J$$

### Diagonal components:

$$C_{II} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_I \quad \text{(no sum)}$$

1st Cauchy-Green: $\underline{\lambda}(\underline{Y}) = \sqrt{\underline{Y} \cdot \underline{\underline{C}}\, \underline{Y}}$

$$\Rightarrow \quad C_{II} = \lambda^2(\underline{e}_I) \quad \checkmark$$

## Off-diagonal components

2nd Cauchy-Green: $\cos\theta(\underline{e}_I, \underline{e}_J) = \dfrac{\underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_J}{\sqrt{\underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_I}\sqrt{\underline{e}_J \cdot \underline{\underline{C}}\, \underline{e}_J}}$

substitute $\quad C_{IJ} = \underline{e}_I \cdot \underline{\underline{C}}\, \underline{e}_J$

$$\Rightarrow \quad C_{IJ} = \lambda(\underline{e}_I)\, \lambda(\underline{e}_J) \cos\theta(\underline{e}_I, \underline{e}_J) \; .$$

The shear between two basis vectors is

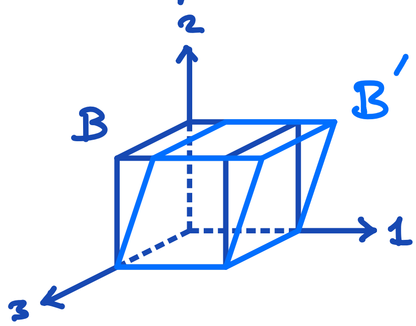
$$\gamma(\underline{e}_I, \underline{e}_J) = \underbrace{\Theta(\underline{e}_I, \underline{e}_J)}_{\frac{\pi}{2}} - \theta(\underline{e}_I, \underline{e}_J)$$

$$\theta(\underline{e}_I, \underline{e}_J) = \frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)$$

so that
$$C_{IJ} = \lambda(\underline{e}_I)\, \lambda(\underline{e}_J) \cos\left(\frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)\right)$$
$$= \lambda(\underline{e}_I)\, \lambda(\underline{e}_J) \sin\left(\gamma(\underline{e}_I, \underline{e}_J)\right) \quad \checkmark$$

The components of $\underline{\underline{C}}$ directly quantify stretch and shear unlike the components of $\underline{\underline{F}}$.

# Example: Simple shear




$$B = \{ \underline{X} \in \mathbb{E}^3 \mid 0 < X_i < 1 \}$$

$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} X_1 + \alpha X_2 \\ X_2 \\ X_3 \end{bmatrix} \qquad \alpha > 0$$

"simple shear in $\underline{e}_1$-$\underline{e}_2$ plane

Deformation gradient:

$$[\underline{\underline{F}}] = [\nabla \underline{\varphi}] = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ homogeneous deformation

Cauchy-Green strain tensor:

$$[\underline{\underline{C}}] = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1+\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

Find the shear $\gamma$ for direction pair $(\underline{e}_1, \underline{e}_2)$

$$\gamma(\underline{e}_1, \underline{e}_2) = \Theta(\underline{e}_1, \underline{e}_2) - \theta(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \theta(\underline{e}_1, \underline{e}_2)$$

$$\cos\theta(\underline{e}_1, \underline{e}_2) = \frac{[\underline{e}_1]^T [\underline{\underline{C}}][\underline{e}_2]}{\sqrt{[\underline{e}_1]^T [\underline{\underline{C}}][\underline{e}_1]}\sqrt{[\underline{e}_2]^T [\underline{\underline{C}}][\underline{e}_2]}} = \frac{\alpha}{\sqrt{1}\sqrt{1+\alpha^2}}$$

$$\Rightarrow \underline{\underline{\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \operatorname{acos}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)}}$$

Find $\gamma(\underline{e}_1, \underline{e}_3)$ again $\Theta(\underline{e}_1, \underline{e}_3) = \frac{\pi}{2}$

$$\cos\Theta(\underline{e}_1, \underline{e}_3) = \frac{C_{13}}{\sqrt{C_{11}}\,\sqrt{C_{33}}} = \frac{0}{1 \cdot 1} = 0$$

$$\gamma(\underline{e}_1, \underline{e}_3) = \frac{\pi}{2} - \underbrace{\operatorname{acos} 0}_{\frac{\pi}{2}} = \underline{\underline{0}}$$

What are the extreme values of the strech and their directions? $\Rightarrow$ eigenvalues & vectors

$$\begin{vmatrix} 1-\lambda^2 & \alpha & 0 \\ \alpha & 1+\alpha^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{vmatrix} = 0$$

$$\lambda_1^2 = 1 + \frac{\alpha^2}{2} + \alpha\sqrt{1+\alpha^2/4} > 1$$
$$\lambda_2^2 = 1$$
$$\lambda_3^2 = 1 + \frac{\alpha^2}{2} - \alpha\sqrt{1+\alpha^2/4} < 1$$

Principal directions:

$$[\underline{v}_1] = [\sqrt{1+\alpha^2/4} - \alpha/2, 1, 0]$$

$$[\underline{v}_2] = [0, 0, 1]$$ (not normalized)

$$[\underline{v}_3] = [\sqrt{1+\alpha^2/4} + \alpha/2, -1, 0]$$

$\Rightarrow$ $\lambda_1$ is max strech in dir $\underline{v}_1$

$\lambda_3$ is min strech in dir $\underline{v}_3$

there is no strech in dir $\underline{e}_3$