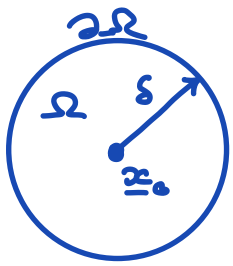
Cauchy Stress Tensor

Force balance in limit of small body.



$$\lim_{\delta \to 0} \underline{f} = m\underline{a}$$

$$m = \int_\Omega \rho \, dV$$

$$\underline{f} = \underline{F}_b[\Omega] + \underline{F}_s[\partial\Omega]$$

$$= \int_\Omega \underline{b} \, dV + \oint_{\partial\Omega} \underline{t} \, dS$$

V_Ω = volume

$A_{\partial\Omega}$ = surf. area

$$\lim_{\delta \to 0} m = \int_\Omega \rho(\underline{x}) \, dV \approx \rho(\underline{x}_0) \int_\Omega dV = \rho_0 V_\Omega$$

$$\lim_{\delta \to 0} \underline{F}_b[\Omega] = \int_\Omega \underline{b} \, dV \approx \underline{b}(\underline{x}_0) \int_\Omega dV = \underline{b}_0 V_\Omega$$

substitute into 2nd law

$$\lim_{\delta \to 0} \oint_{\partial\Omega} \underline{t} \, dA = \int_\Omega \rho\underline{a} - \underline{b} \, dV \approx (\rho\underline{a} - \underline{b}) V_\Omega$$

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega}} \oint_{\partial\Omega} \underline{t} \, dA = \frac{V_\Omega}{A_{\partial\Omega}} (\rho\underline{a} - \underline{b})$$

Volume vanishes faster than surface area!

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_{\partial\Omega}} = 0$$

Consider a shere: $V_\Omega = \frac{4}{3}\pi\delta^3$ $A_{\partial\Omega} = 4\pi\delta^2$

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_{\partial\Omega}} = \frac{\delta}{3} = 0$$

But this holds for any shape.

HW: Show $\frac{V}{A} \to 0$ for tetrahedron

Surface forces vanish of infinitesimal body

$$\boxed{\lim_{\delta \to 0} \frac{1}{A_\Omega} \oint_{\partial\Omega} \underline{t}\, dA = 0}$$

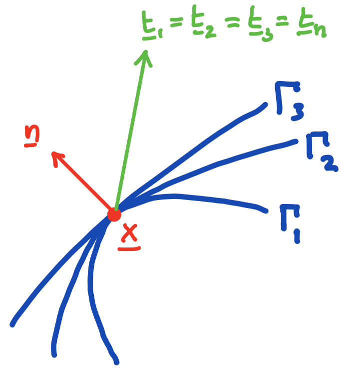
Note: • $\frac{1}{A_\Omega}$ normalization

• assume ρ, $|\underline{g}|$ and $|\underline{a}|$ are finite

$\Rightarrow$ basis for derivation of Cauchy stress tensor.

Cauchy's postulate

The traction field $\underline{t}_n$ on a surface Γ in B depends only pointwise on the unit normal field $\underline{n}$. In particular, there is a traction function such that $\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}), \underline{x})}$.



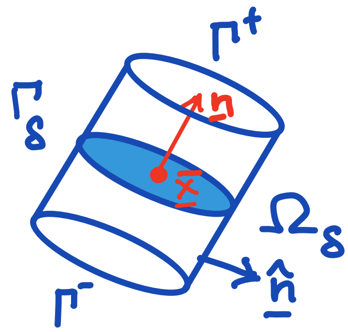
This assumes that the traction field is independent of $\nabla \underline{n}$ and hence the curvature of the surface. Therefore the traction $\underline{t}_i$ on the set of surfaces Γ_i that are tangent at $\underline{x}$ is the same, $\underline{t}_i = \underline{t}_n$.

Law of Action and Reaction

If the traction field, $\underline{t}(\underline{n},\underline{x})$, is continuous and bounded, then

$$\boxed{\underline{t}(-\underline{n},\underline{x}) = -\underline{t}(\underline{n},\underline{x})}$$

for all $\underline{n}$ and $\underline{x} \in B$.



Disk D centered on $\underline{x}$

Cylinder Ω_δ center $\underline{x}$ axis $\underline{n}$.

Faces: $\Gamma^+, \Gamma^-, \Gamma_\delta$

outward normal $\hat{\underline{n}}$ on $\partial\Omega_\delta = \Gamma^+ \cup \Gamma^- \cup \Gamma_\delta$

union

Note: on Γ^+ $\hat{\underline{n}} = \underline{n}$ and on Γ^- $\hat{\underline{n}} = -\underline{n}$

$\delta \to 0$: $\Gamma^\pm \to D$ and $\Gamma_\delta \to 0$

Resultant surface force

$$\underline{r}_s[\partial\Omega_\delta] = \int_{\partial\Omega_\delta} \underline{t}_n(\underline{n},\underline{x})\, dA$$

since $\quad \partial\Omega_\delta = \Gamma_\delta \cup \Gamma^+ \cup \Gamma^-$

$$\underline{r}_s[\partial\Omega_\delta] = \int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}},\underline{x})\, dA + \int_{\Gamma^+} \underline{t}(\underline{n},\underline{x})\, dA + \int_{\Gamma^-} \underline{t}(-\underline{n},\underline{x})\, dA$$

$$\lim_{\delta\to 0}\left[\int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}}(\underline{x}),\underline{x})\, dA + \int_{\Gamma^+} \underline{t}(\underline{n},\underline{x})\, dA + \int_{\Gamma^-} \underline{t}(-\underline{n},\underline{x})\, dA\right] = 0$$

the first term vanishes because $\underline{t}$ is bounded and $\Gamma_\delta \to 0$. Using the fact that $\Gamma^\pm \to D$ we have in the limit

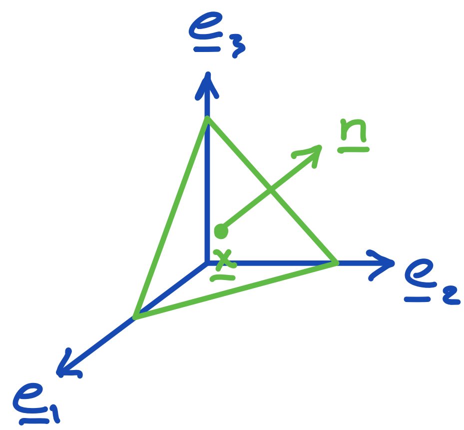
$$\int_D \underline{t}(\underline{n},\underline{y}) + t(-\underline{n},\underline{y})\, dA = 0$$

since the radius of D is arbitrary the integrand must vanish so that $\quad \underline{t}(\underline{n},\underline{x}) + \underline{t}(-\underline{n},\underline{x}) = 0$ ✓

The Stress tensor

Cauchy's Theorem

Let $\underline{t}(\underline{n},\underline{x})$ be the traction field for body B that satisfies Cauchy's postulate. Then $\underline{t}(\underline{n},\underline{x})$ is linear in $\underline{n}$, that is, for each $\underline{x}\in B$ there is a second-order tensor field $\underline{\underline{\sigma}}(\underline{x})\in \mathcal{V}^2$ such that $\boxed{\underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$ called the Cauchy stress field for B.



To estabish this consider a frame $\{\underline{e}_i\}$, a point $\underline{x}\in B$ and a normal $\underline{n}$ s.t. $\underline{n}\cdot\underline{e}_i > 0$.

For $\delta > 0$, let Γ_δ denote a triangular region with center $\underline{x}$, normal $\underline{n}$ and maximum edge length δ.

Let Ω_δ be the tetrahedron bounded by Γ_δ and the three coordinate planes. These planes form three faces Γ_j with outward normals $\underline{n}_j = -\underline{e}_j$. The volume of Ω_δ goes to zero as δ becomes small.

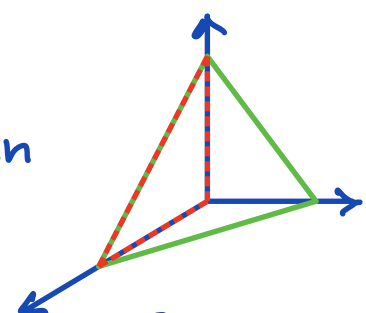
$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} \underline{t}(\underline{n}(\underline{y}), \underline{y})\, dA = 0$$

where $A_{\partial\Omega_\delta}$ is the surface area of Ω_δ

Since $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we have

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j, \underline{y})\, dA \right] = 0$$

Since each face Γ_j can be linearly mapped onto Γ_δ with constant Jacobian $n_j = \underline{n} \cdot \underline{e}_j > 0$ so that $A_{\Gamma_j} = n_j A_{\Gamma_\delta}$



$$\Rightarrow A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=1}^{3} \Gamma_j = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substituting we obtain

$$\lim_{\delta \to 0} \frac{1}{A_{\partial \Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\delta} t_n(-\underline{e}_j, \underline{y})\, n_j \, dA \right] = 0$$

$$\lim_{\delta \to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y}) + \sum_{j=1}^{3} t(-\underline{e}_j, \underline{y})\, n_j \, dA = 0$$

As $\delta \to 0$ the area Γ_δ shrinks to $\underline{x}$ so that by the mean value Thm for integrals the limit is given by the integrand. Hence

$$\underline{t}(\underline{n}, \underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j = 0$$

Using the Law of Action and Reaction

$$\underline{t}(\underline{n}, \underline{x}) = -\sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j = \sum_{j=1}^{3} \underline{t}(\underline{e}_j, \underline{x})\, n_j$$

or with summation convention

$$\underline{t}(\underline{n}, \underline{x}) = \underline{t}(\underline{e}_j, \underline{x})\, n_j$$

using the definition of dyadic product

$$\left(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j\right)\underline{n} = \underbrace{(\underline{e}_j \cdot \underline{n})}_{n_i\, \underline{e}_j \cdot \underline{e}_i \,=\, n_i\, \delta_{ij} \,=\, n_j}\, \underline{t}(\underline{e}_j, \underline{x})$$

So that we have

$$\underline{t}(\underline{n}, \underline{x}) = \left(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j\right)\underline{n} = \underline{\underline{\sigma}}\, \underline{n}$$

$$\underline{\underline{\sigma}} = \underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j$$

substituting $\underline{t}(\underline{e}_j, \underline{x}) = t_i(\underline{e}_j, \underline{x})\, \underline{e}_i$ we obtain the definition of the Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j \quad \text{with} \quad \sigma_{ij} = t_i(\underline{e}_j, \underline{x})}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.

The traction vectors on the coor. planes at $\underline{x}$ are

$$\underline{t}(\underline{e}_1, \underline{x}) = t_i(\underline{e}_1, \underline{x})\, \underline{e}_i = S_{i1}(\underline{x})\, \underline{e}_i$$

$$\underline{t}(\underline{e}_2, \underline{x}) = t_i(\underline{e}_2, \underline{x})\, \underline{e}_i = S_{i2}(\underline{x})\, \underline{e}_i$$

$$\underline{t}(\underline{e}_3, \underline{x}) = t_i(\underline{e}_3, \underline{x})\, \underline{e}_i = S_{i3}(\underline{x})\, \underline{e}_i$$