

Change in basis

Both $\underline{v} \in \mathcal{V}$ and $\underline{s} \in \mathcal{V}^2$ are invariant upon change of basis, but their representations $[\underline{v}]$ and $[\underline{s}]$ change.

Consider two frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

Representation of \underline{e}'_j in $\{\underline{e}_i\}$ is

$$\begin{aligned}\underline{e}'_j &= (\underline{e}'_j \cdot \underline{e}_1) \underline{e}_1 + (\underline{e}'_j \cdot \underline{e}_2) \underline{e}_2 + (\underline{e}'_j \cdot \underline{e}_3) \underline{e}_3 \\ &= (\underline{e}'_j \cdot \underline{e}_i) \underline{e}_i\end{aligned}$$

$$\underline{e}'_j = A_{ij} \underline{e}_i \quad \text{where } A_{ij} = \underline{e}_i \cdot \underline{e}'_j$$

↑
note transpose

Here \underline{A} is the change of basis tensor

$$\underline{A} = A_{ij} \underline{e}_i \otimes \underline{e}_j \quad A_{ij} = \underline{e}_i \cdot \underline{e}'_j$$

Similarly we can express \underline{e}_i in $\{\underline{e}'_k\}$

$$\underline{e}_i = (\underline{e}_i \cdot \underline{e}'_k) \underline{e}'_k = A_{ik} \underline{e}'_k$$

We have $\underline{e}'_j = A_{ij} \underline{e}_i$ $\underline{e}_i = A_{ik} \underline{e}'_k$

$$\left. \begin{aligned} \underline{e}'_j &= A_{ij} A_{ik} \underline{e}'_k \\ \underline{e}'_j &= \underbrace{A_{ij} A_{ik}}_{\delta_{jk}} \underline{e}'_k \end{aligned} \right\} \begin{aligned} \underline{e}_i &= A_{ik} (A_{lk} \underline{e}_l) \\ &= A_{ik} A_{lk} \underline{e}_l \\ \underline{e}_i &= \delta_{il} \underline{e}_l \\ A_{ik} A_{lk} &= \delta_{il} \end{aligned}$$

$$A_{ij} A_{ik} = \delta_{ik}$$

\Rightarrow $\underline{A}^T \underline{A} = \underline{A} \underline{A}^T = \underline{I}$ \underline{A} is orthogonal

If both $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ are right-handed
then \underline{A} must be a rotation, $\det(\underline{A}) = 1$

Change in representation

Consider $\underline{v} \in \mathcal{V}$ and $\underline{s} \in \mathcal{V}^2$

with representations:

$$[\underline{v}] \text{ and } [\underline{s}] \text{ in } \{\underline{e}_i\}$$

$$[\underline{v}]' \text{ and } [\underline{s}]' \text{ in } \{\underline{e}'_i\}$$

so that $[\underline{v}] \neq [\underline{v}]'$ and $[\underline{s}] \neq [\underline{s}]'$

then $\underline{[v]} = \underline{[A]} \underline{[v]}'$ and $\underline{[v]}' = \underline{[A]}^T \underline{[v]}$

to see this

$$\underline{v} = v_i \underline{e}_i = v_j' \underline{e}_j \quad \text{where} \quad \underline{e}_j' = A_{ij} \underline{e}_i$$
$$v_i \underline{e}_i = v_j' A_{ij} \underline{e}_i \Rightarrow v_i = A_{ij} v_j' \quad \checkmark$$

Similarly

$$\underline{[s]} = \underline{[A]} \underline{[s]}' \underline{[A]}^T$$
$$\underline{[s]}' = \underline{[A]}^T \underline{[s]} \underline{[A]}$$

\Rightarrow HW2

In summary; Change of basis is a rotation and can hence be written as an orthogonal tensor \underline{A} with components $A_{ij} = \underline{e}_i \cdot \underline{e}_j$.

Next we show that $\text{tr}(\underline{s})$ and $\det(\underline{s})$ are invariant under change in basis
 \Rightarrow important for constitutive laws

Invariance of trace

For $\underline{S} \in \mathcal{V}^2$ with $[\underline{S}]$ in $\{e_i\}$ and $[\underline{S}]'$ in $\{e'_i\}$

$$\boxed{\text{tr}[\underline{S}] = \text{tr}[\underline{S}]'}$$

considers

$$[\underline{S}] = [\underline{A}][\underline{S}]'[\underline{A}]^T \quad \text{or} \quad [\underline{S}]_{ij} = [\underline{A}]_{ik} [\underline{S}]'_{kl} [\underline{A}]_{jl}$$

$$\begin{aligned} \text{tr}[\underline{S}] &= [\underline{S}]_{ii} = [\underline{A}]_{ik} [\underline{S}]'_{kl} [\underline{A}]_{il} \\ &= [\underline{S}]'_{kl} \underbrace{[\underline{A}]_{ik} [\underline{A}]_{il}}_{\delta_{kl}} = [\underline{S}]'_{kk} = \text{tr}[\underline{S}]' \end{aligned}$$

Invariance of determinant

For $\underline{S} \in \mathcal{V}^2$ with $[\underline{S}]$ in $\{e_i\}$ and $[\underline{S}]'$ in $\{e'_i\}$

$$\boxed{\det[\underline{S}] = \det[\underline{S}]'} \Rightarrow \text{HW 2}$$

Eigenvalues & vectors of tensors

By the eigen pair of $\underline{\underline{S}} \in \mathcal{V}^2$ we mean the scalar λ and the vector \underline{v} such that

$$\underline{\underline{S}} \underline{v} = \lambda \underline{v}$$

$\lambda =$ eigenvalue $\underline{v} =$ eigenvector

λ 's are roots of characteristic polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p we one or more \underline{v}_p satisfying

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}}) \underline{v}_p = \underline{0}$$

In continuum mechanics we are mostly concerned with symmetric tensors, eg. stress.

Eigen problem for symmetric tensors

1) All λ_p 's real

2) All λ_p 's are positive ($\underline{\underline{S}}$ sym. pos. def.)

3) All \underline{v}_p 's corresponding to distinct λ_p 's are orthogonal

$\underline{\underline{S}}$ is symmetric positive definite (spd)

if $\underline{v} \cdot \underline{\underline{S}} \underline{v} > 0$ for all $\underline{v} \in \mathcal{V}$

by def. of eigenpair $\underline{\underline{S}} \underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot \lambda \underline{v} > 0$$

$$\lambda |\underline{v}|^2 > 0 \rightarrow \lambda > 0$$

For orthogonality consider two eigen pairs

(λ, \underline{v}) and (ω, \underline{u}) so that $\lambda \neq \omega$

$$\underline{S} \underline{v} = \lambda \underline{v} \quad \text{and} \quad \underline{S} \underline{u} = \omega \underline{u}$$

$$\begin{aligned} \text{Consider } \lambda (\underline{v} \cdot \underline{u}) &= \underline{S} \underline{v} \cdot \underline{u} = \underline{v} \cdot \underline{S}^T \underline{u} \quad (\underline{S}^T = \underline{S}) \\ &= \underline{v} \cdot \underline{S} \underline{u} = \omega (\underline{v} \cdot \underline{u}) \end{aligned}$$

$$\lambda (\underline{v} \cdot \underline{u}) = \omega (\underline{v} \cdot \underline{u})$$

$$\text{since } \lambda \neq \omega \Rightarrow \underline{v} \cdot \underline{u} = 0$$

Spectral decomposition

If $\underline{S} = \underline{S}^T$ there exists a frame $\{\underline{e}_i\}$ consisting of the eigenvectors of \underline{S} so that

$$\underline{S} = \sum_{i=1}^3 \lambda_i \underline{v}_i \otimes \underline{v}_i$$

To see this consider the following

Since \underline{v}_i are orthonormal: $\underline{I} = \underline{v}_i \otimes \underline{v}_i$

$$\begin{aligned}\underline{\underline{A}} &= \underline{\underline{A}} \underline{\underline{I}} = \underline{\underline{A}} (\underline{v}_i \otimes \underline{v}_i) = (\underline{\underline{A}} \underline{v}_i) \otimes \underline{v}_i = \sum_{i=1}^3 (\lambda_i \underline{v}_i) \otimes \underline{v}_i \\ &= \sum_{i=1}^3 \lambda_i \underline{v}_i \otimes \underline{v}_i\end{aligned}$$

where we have used: $\underline{\underline{A}}(\underline{u} \otimes \underline{v}) = (\underline{\underline{A}} \underline{u}) \otimes \underline{v} \Rightarrow \text{HW 2}$

$$\alpha(\underline{u} \otimes \underline{v}) = (\alpha \underline{u}) \otimes \underline{v}$$

Representation in eigenframe

$$[\underline{\underline{S}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{diagonalize tensor}$$

The principal invariants of $\underline{\underline{S}} \in \mathcal{V}^2$ are

$$I_1(\underline{\underline{S}}) = \text{tr}(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2(\underline{\underline{S}}) = \frac{1}{2} \left((\text{tr} \underline{\underline{S}})^2 - \text{tr}(\underline{\underline{S}}^2) \right) = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$$

$$I_3(\underline{\underline{S}}) = \det(\underline{\underline{S}}) = \lambda_1 \lambda_2 \lambda_3$$

These 3 scalars are frame invariant.

Set of invariants $I_S = \{I_i(\underline{\underline{S}})\}$

Applications of Invariants

The first invariant of stress tensor simply shows that the mean normal stress (pressure) is independent of frame.

The second invariant is important in theories of creep / Non-Newtonian flows

The second and third invariant are important in theories of plastic yield.

Rewrite the char. polynomial with invariants

$$\det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = -\lambda^3 + I_1(\underline{\underline{S}})\lambda^2 - I_2(\underline{\underline{S}})\lambda + I_3(\underline{\underline{S}}) = 0$$

This is most easily shown in eigen frame just collect terms in powers of λ .

Consider $\underline{\underline{S}} \underline{\underline{S}} \underline{\underline{v}} = \underline{\underline{S}} (\lambda \underline{\underline{v}}) = \lambda^2 \underline{\underline{v}}$

in general $\underline{\underline{S}}^a \underline{\underline{v}} = \lambda^a \underline{\underline{v}}$

multiplying char. poly. by $\underline{\underline{v}}$

$$-\lambda^3 \underline{\underline{v}} + I_1(\underline{\underline{S}}) \lambda^2 \underline{\underline{v}} - I_2(\underline{\underline{S}}) \lambda \underline{\underline{v}} + I_3(\underline{\underline{S}}) \underline{\underline{v}} = 0$$

$$-\underline{\underline{S}}^3 \underline{\underline{v}} + I_1(\underline{\underline{S}}) \underline{\underline{S}}^2 \underline{\underline{v}} - I_2(\underline{\underline{S}}) \underline{\underline{S}} \underline{\underline{v}} + I_3(\underline{\underline{S}}) \underline{\underline{v}} = 0$$

since $\underline{\underline{v}} \neq 0$

$$-\underline{\underline{S}}^3 + I_1(\underline{\underline{S}}) \underline{\underline{S}}^2 - I_2(\underline{\underline{S}}) \underline{\underline{S}} + I_3(\underline{\underline{S}}) = 0$$

every tensor satisfies its char. poly.

Cayley-Hamilton Theorem

Tensor square root

If $\underline{\underline{C}}$ is a s.p.d. tensor with ei enpair $(\lambda, \underline{\underline{v}})$

then there is a unique tensor $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$

$$\underline{\underline{U}} = \sum_{i=1}^3 \sqrt{\lambda_i} \underline{\underline{e}}_i \otimes \underline{\underline{e}}_i$$