Constitutive Theory

Common constitutive laws:

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta\,(\nabla\underline{v} + \nabla\underline{v}^T)$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ $\quad \eta$ = viscosity $\quad \underline{v}$ = velocity

Linear elastic solid: $\underline{\underline{\sigma}} = \lambda\,\nabla\cdot\underline{u}\,\underline{\underline{I}} + \mu\,(\nabla\underline{u} + \nabla\underline{u}^T)$

λ, μ = lame parameters $\quad \underline{u}$ = displacement

Both derive from the functional form

$$\boxed{\underline{\underline{G}}(\underline{\underline{E}}) = \lambda\,\mathrm{tr}(\underline{\underline{E}}) + 2\mu\,\mathrm{sym}(\underline{\underline{E}})}$$

Newtonian fluid: $\underline{\underline{E}} = \nabla\underline{v}$

Linear elastic solid: $\underline{\underline{E}} = \nabla\underline{u}$

remember $\quad \nabla\cdot\underline{a} = \mathrm{tr}(\nabla\underline{a})$

$\Rightarrow$ direct for lin. elastic solid

for fluid there is a complication due to incompressibility !

Why do const. relations have this form ?

Change of observer

In Lecture 6 we discussed Change in basis

$$\underline{v} = \underline{\underline{Q}}\,\underline{v}' \quad \text{and} \quad \underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$$

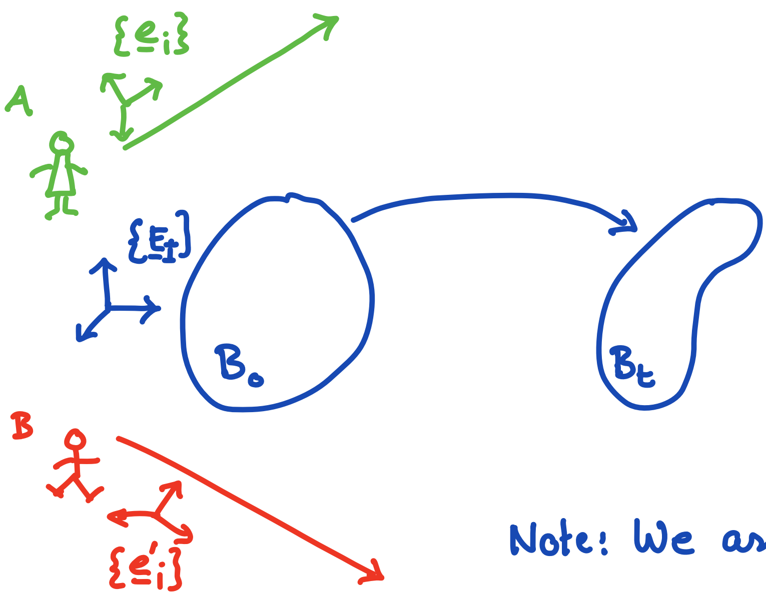
where $\underline{\underline{Q}}$ is change in basis tensor.

Q is a rotation : 1) orthonormal $\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}}$

2) $\det(\underline{\underline{Q}}) = 1$

Change in basis is passive change of frame.

Active change in frame → change in observer



$$\underline{x} = \varphi(\underline{X}, t)$$

$$\underline{x}' = \varphi'(\underline{X}, t)$$

Note: Material ref. frame is comon

Note: We assume both observers use same clock.

Since change in observer cannot induce a deformation. Two ref. frames must be related by a rigid body motion.

$$\boxed{\underline{x}' = Q(t)\,\varphi(\underline{X},t) + \underline{c}(t)} \quad \text{Eulerian transformation}$$

$\underline{\underline{Q}}$ = rotation $\quad\quad$ $\underline{c}$ = translation

Our description of forces and deformations cannot depend on the observer (objective).

Effect on kinematic quantities

$\underline{x} = \varphi(\underline{X},t) \quad\quad \nabla\varphi = \underline{\underline{F}}$

$\underline{x}' = \varphi'(\underline{X},t) = Q\,\varphi(\underline{X},t) + c \quad\quad \nabla\varphi' = \underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}'$

Right Cauchy-Green Strain tensor

$$\underline{\underline{C}}' = \underline{\underline{F}}'^T\,\underline{\underline{F}}' = (\underline{\underline{Q}}\,\underline{\underline{F}})^T(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow$ not affected by rigid body motion because it is a material tensor C_{IJ} (naturally objective)

What about <u>spatial</u> tensors?

Axiom of frame indifference

Fields ϕ, $\underline{w}$ and $\underline{\underline{S}}$ are called frame indifferent or objective if for all superposed rigid body motions $\underline{x}' = \underline{\underline{Q}}\,\underline{x} + \underline{c}$ we have for all spatial fields

$$\boxed{\begin{aligned} \phi'(\underline{x}', t) &= \phi(\underline{x}, t) \\ \underline{w}'(\underline{x}', t) &= \underline{\underline{Q}}\,\underline{w}(\underline{x}, t) \\ \underline{\underline{S}}'(\underline{x}', t) &= \underline{\underline{Q}}\,\underline{\underline{S}}(\underline{x}, t)\,\underline{\underline{Q}}^T \end{aligned}}
\quad \begin{aligned} &\text{scalar field} \\ &\text{vector field} \\ &\text{tensor field} \end{aligned}$$

$\Rightarrow$ from Lecture 6.

Is spatial velocity gradient objective?

From lecture 16: $\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$ $\qquad \underline{\underline{\ell}}' = \nabla_{x'}' \underline{v}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1}$

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$$

$\Rightarrow \underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ <u>not</u> objective!

that is why $\nabla_x \underline{v}$ is not used in constitutive laws

The "non-objective" term is $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

it represents rigid body angular velocity between observers. see HW9

Show $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ skew-symmetric

Non-objetive part of $\underline{\underline{\ell}} = \nabla_x \underline{v}$ is skew-sym.

$\Rightarrow$ simply take symmetric part of $\underline{\underline{\ell}}$!

$\underline{\underline{d}} = \text{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws

Note that velocity itself

Material frame indifferent functions

Fields: $\phi(\underline{x},t)$ scalar

$\underline{w}(\underline{x},t)$ vector

$\underline{\underline{s}}(\underline{x},t)$ tensor

fields because they depend on $\underline{x}$.

Constitutive functions are not fields but they depend on fields as input.

internal energy: $\underset{\text{output field}}{u(\underline{x},t)} = \underset{\text{constitutive function}}{\hat{u}}(\underbrace{\rho(\underline{x},t), \theta(\underline{x},t)}_{\text{input fields}})$

heat flow: $\underline{q}(\underline{x},t) = \hat{\underline{q}}(\theta(\underline{x},t))$

Cauchy stress: $\underline{\underline{s}}(\underline{x},t) = \hat{\underline{\underline{s}}}(\rho(\underline{x},t), \theta(\underline{x},t), \underline{\underline{d}}(\underline{x},t))$

Constitutive functions: $\hat{u}(\rho,\theta)$, $\hat{\underline{q}}(\theta)$, $\hat{\underline{\underline{s}}}(\rho,\theta,\underline{\underline{d}})$

As such constitutive functions are not directly dependent on frame but their input fields are.

Consider frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ then to be frame indifference requires

$$\hat{\underline{\underline{\sigma}}}(\rho', \theta', \underline{\underline{d}}') = \underline{\underline{Q}}\ \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}})\ \underline{\underline{Q}}^T$$

substituting $\underline{\underline{d}}' = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$

$$\boxed{\hat{\underline{\underline{\sigma}}}(\rho', \theta', \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\ \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{d}})\ \underline{\underline{Q}}^T}$$

$\Rightarrow$ both input & output of constitutive function $\hat{\underline{\underline{\sigma}}}$ must be frame invariant

Isotropic functions

Functions that are frame invariant are called isotropic. Consider the following

$\hat{\phi}$ = scalar fun. $\quad \hat{\underline{w}}$ = vector fun. $\quad \hat{\underline{\underline{\sigma}}}$ = tensor fun.

θ = scalar $\quad \underline{v}$ = vector $\quad \underline{\underline{S}}$ = tensor

Then for two frames related by rigid body rotation $\underline{\underline{Q}}$ we have following isotropic functions:

$$\boxed{\begin{array}{lll}
\hat{\phi}(\theta) = \hat{\phi}(\theta) & \hat{\phi}(\underline{\underline{Q}}\,\underline{v}) = \hat{\phi}(\underline{v}) & \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\underline{\underline{S}}) \\
\hat{\underline{w}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{w}}(\theta) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{v}) & \hat{\underline{w}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{w}}(\underline{\underline{S}}) \\
\hat{\underline{\underline{\sigma}}}(\theta) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\theta)\,\underline{\underline{Q}}^T & \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{v})\,\underline{\underline{Q}}^T & \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T
\end{array}}$$

Examples:

1) $\hat{\phi}(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{S}})\det(\underline{\underline{Q}}^T) = \det(\underline{\underline{S}}) \checkmark$$

2) $\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\hat{\underline{u}}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T\,\underline{\underline{Q}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{Q}}\,\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) \checkmark$$

Isotropic material: stress/strain principal directions

Objectivity $\Rightarrow$ isotropic function

fluids: $\underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{Q}}^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T)$ rate of strain

solids: $\underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{\varepsilon}})\,\underline{\underline{Q}}^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{Q}}^T)$ strain

generic: $\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T = \underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T)$

Since $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ and $\underline{\underline{d}} = \underline{\underline{d}}^T$ ($\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^T$) they can all be written in spectral decomposition.

$$\underline{\underline{S}} = \underline{\underline{S}}^T \quad \Rightarrow \quad \underline{\underline{S}} = \sum_{i=1}^{3} \alpha_i \, \underline{v}_i \otimes \underline{v}_i$$

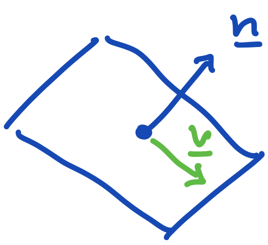
where $\underline{\underline{S}}\,\underline{v}_i = \alpha_i \underline{v}_i$

Q: How are $\underline{v}_i$'s of $\underline{\underline{\sigma}}$ and $\underline{\underline{d}}/\underline{\underline{\varepsilon}}$ related?

$\Rightarrow$ same eigenvectors!

$$\underline{\underline{A}}\,\underline{v}_i = \alpha_i \underline{v}_i \quad \Rightarrow \quad \underline{\underline{G}}(\underline{\underline{A}})\,\omega_i = \omega_i \underline{v}_i$$

Can be shown with reflections & projections



Projection: $\underline{\underline{P}}_n = \underline{n} \otimes \underline{n}$

Reflection: $\underline{\underline{R}}_n = \underline{\underline{I}} - 2\,\underline{n} \otimes \underline{n}$

Note: $\underline{\underline{R}}_n \underline{n} = -\underline{n}$

$\underline{\underline{R}}_n \underline{a} = \underline{a}$ if $\underline{a} \cdot \underline{n} = 0$ $\quad \underline{a} \perp \underline{n}$

$\Rightarrow$ reflections help to detect colinear vectors.

If $\underline{n} = \underline{v}_1$ one eigenvector

$\Rightarrow \underline{\underline{R}}_{v_1} \underline{v}_1 = -v_1$ but $\underline{\underline{R}}_{v_1} \underline{v}_2 = \underline{v}_2$ & $\underline{\underline{R}}_{v_1} \underline{v}_3 = \underline{v}_3$

Step 1: $\boxed{\underline{\underline{R}}_{v_1} \underline{\underline{A}}\, \underline{\underline{R}}_{v_1}^T = \underline{\underline{A}}}$ $\qquad \underline{\underline{S}} = \underline{\underline{S}}^T$

$$= \underline{\underline{R}}_{v_1} \left(\sum_{i=1}^{3} \alpha_i (\underline{v}_i \otimes \underline{v}_i) \right) \underline{\underline{R}}_{v_1}^T$$

$$= \sum_{i=1}^{3} \alpha_i\, \underline{\underline{R}}_{v_1} (\underline{v}_i \otimes \underline{v}_i)\, \underline{\underline{R}}_{v_1}^T$$

use identities: $\underline{\underline{S}}(\underline{a} \otimes \underline{b}) = (\underline{\underline{S}}\underline{a}) \otimes \underline{b}$

$$(\underline{a} \otimes \underline{b})\underline{\underline{S}} = \underline{a} \otimes (\underline{\underline{S}}^T \underline{b})$$

$$\Rightarrow \underline{\underline{R}}_{v_1} \underline{\underline{A}}\, \underline{\underline{R}}_{v_1}^T = \sum_{i=1}^{3} \alpha_i (\underline{\underline{R}}_{v_1} \underline{v}_i) \otimes (\underline{\underline{R}}_{v_1} \underline{v}_i)$$

$$= \alpha_1 (-\underline{v}_1) \otimes (-\underline{v}_1) + \alpha_2\, \underline{v}_2 \otimes \underline{v}_2 + \alpha_3\, \underline{v}_3 \otimes \underline{v}_3$$

$$= \sum_{i=1}^{3} \alpha_i\, \underline{v}_i \otimes \underline{v}_i = \underline{\underline{A}}$$

$$\Rightarrow \quad \underline{\underline{R}}_{v_1} \underline{\underline{d}}\, \underline{\underline{R}}_{v_1}^T = \underline{\underline{d}}$$

$$\underline{\underline{R}}_{v_1} \underline{\underline{\varepsilon}}\, \underline{\underline{R}}_{v_1}^T = \underline{\underline{\varepsilon}}$$

Step 2: $\boxed{\underline{\underline{R}}_{v_i}\,\underline{\underline{G}}(\underline{\underline{A}}) = \underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{R}}_{v_i}}$ commute

isotropic material: $\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T = \underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T)$

$\underline{\underline{Q}}$ = orthogonal (rotation or reflection)

$\underline{\underline{Q}} = \underline{\underline{R}}_{v_1}$: $\underline{\underline{R}}_{v_1}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{R}}_{v_1}^T = \underline{\underline{G}}(\underline{\underline{R}}_{v_1}\,\underline{\underline{A}}\,\underline{\underline{R}}_{v_1}^T)$

$$= \underline{\underline{G}}(\underline{\underline{A}})$$

$$\underline{\underline{R}}_{v_1}\,\underline{\underline{G}}(\underline{\underline{A}})\,\overbrace{\cancel{\underline{\underline{R}}_{v_1}^T\,\underline{\underline{R}}_{v_1}}}^{\underline{\underline{I}}} = \underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{R}}_{v_1}$$

$$\Rightarrow \underline{\underline{R}}_{v_1}\,\underline{\underline{G}}(\underline{\underline{A}}) = \underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{R}}_{v_1}$$

Step 3: $\underline{\underline{G}}(\underline{\underline{A}})\,\underline{v}_i = \omega_i\,\underline{v}_i$ where $\underline{\underline{A}}\,\underline{v}_i = \alpha_i \underline{v}_i$

$$\underline{\underline{R}}_{v_i}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{v}_i = \underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{R}}_{v_i}\,\underline{v}_i$$

$$\underline{\underline{R}}_{v_i}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{v}_i = -\underline{v}_i$$

since $\underline{\underline{R}}_{v_i}$ is reflection $\Rightarrow$ $\underline{\underline{G}}(\underline{\underline{A}})\,\underline{v}_i \parallel \underline{v}_i$

$\underline{v}_i$ is only stretched by $\underline{\underline{G}}(\underline{\underline{A}}) \Rightarrow \underline{v}_i$ is an eigenvector!

$$\Rightarrow \boxed{\underline{\underline{G}}(\underline{\underline{A}})\,\underline{v}_i = \omega_i\,\underline{v}_i}$$

principal directions of stress and strain are same

(if material is isotropic)

Proof of Representation Thm (linear, isotropic)

Note: $\underline{\underline{A}} = \sum_{i=1}^{3} \alpha_i \underbrace{\underline{v}_i \otimes \underline{v}_i}_{\underline{\underline{P}}_{v_i}} = \sum_{i=1}^{3} \alpha_i \underline{\underline{P}}_{v_i}$

where $\underline{\underline{P}}_{v_i} = \underline{v}_i \otimes \underline{v}_i$ projection tensor

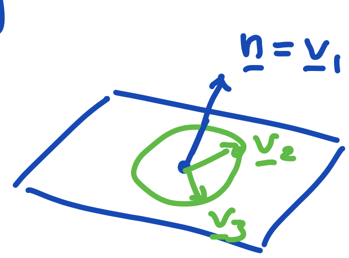
Spectral decomposition of stress response

$$\underline{\underline{G}}(\underline{\underline{A}}) = \underline{\underline{G}}\left(\sum_{i=1}^{3} \alpha_i \underline{\underline{P}}_{v_i}\right) = \sum_{i=1}^{3} \alpha_i \underline{\underline{G}}(\underline{\underline{P}}_{v_i})$$

$\Rightarrow$ total stress response is sum of stress response in principal directions $\underline{\underline{G}}(\underline{\underline{P}}_{v_i})$

Eigenproblem for Projection tensors

$\underline{\underline{P}}_n \beta_i = \beta_i v_i \qquad (\beta_i, \underline{v}_i)$



$\beta_1 = 1 \quad \beta_2 = \beta_3 = 0$

$\Rightarrow \underline{v}_1 = \underline{n}$

$\underline{v}_2$ & $\underline{v}_3$ are any two perpendicular vectors in plane

For any $\underline{\underline{S}} = \underline{\underline{S}}^T$ with $\underline{v}_1$ $\underline{v}_2$ $\underline{v}_3$ indep. β_1 & $\beta = \beta_2 = \beta_3$

$$\boxed{\underline{\underline{S}} = \beta\, \underline{\underline{I}} + (\beta_1 - \beta)\, \underline{\underline{P}}_{v_1}}$$

From spectral decomposition:

$$\underline{\underline{E}} = \beta_1 \underline{\underline{P}}_{v_1} + \beta\, \underline{\underline{P}}_{v_2} + \beta\, \underline{\underline{P}}_{v_3}$$

$$= \beta_1 \underline{\underline{P}}_{v_1} - \beta\, \underline{\underline{P}}_{v_1} + \beta\, \underline{\underline{P}}_{v_1} + \beta\, \underline{\underline{P}}_{v_2} + \beta\, \underline{\underline{P}}_{v_3}$$

$$= (\beta_1 - \beta)\, \underline{\underline{P}}_{v_1} + \beta\, (\underbrace{\underline{\underline{P}}_{v_1} + \underline{\underline{P}}_{v_2} + \underline{\underline{P}}_{v_3}}_{\underline{\underline{I}}})$$

$$\underline{\underline{E}} = \beta\, \underline{\underline{I}} + (\beta_1 - \beta)\, \underline{\underline{P}}_{v_1}$$

Apply to $\underline{\;}(\underline{\underline{P}}_{v_i})$ which as same $\underline{v}_i$s as $\underline{\underline{P}}_{v_i}$

$$\underline{\underline{G}}(\underline{\underline{P}}_{v_i}) = \beta\, \underline{\underline{I}} + (\beta_1 - \beta)\, \underline{\underline{P}}_{v_1}$$

$$= \lambda(\underline{v}_i)\, \underline{\underline{I}} + 2\mu(\underline{v}_i)\, \underline{\underline{P}}_{v_1}$$

we can show $\lambda(\underline{v}_1) = \lambda(\underline{v}_2) = \lambda(\underline{v}_3) = \lambda = \text{const.}$

see HW $\mu(\underline{v}_1) = \mu(\underline{v}_2) = \mu(\underline{v}_3) = \mu = \text{const.}$

$\Rightarrow$ $\underline{\underline{G}}(\underline{\underline{P}}_{v_i}) = \lambda\, \underline{\underline{I}} + 2\mu\, \underline{\underline{P}}_{v_i}$ for any $\underline{v}_i$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \underline{\underline{G}}\left(\sum_{i=1}^{3} \alpha_i \underline{\underline{P}}_{v_i}\right) = \sum_{i=1}^{3} \alpha_i \underline{\underline{G}}(\underline{\underline{P}}_{v_i})$$

$$= \sum_{i=1}^{3} \alpha_i (\lambda \underline{\underline{I}} + 2\mu \underline{\underline{P}}_{v_i})$$

$$= \lambda \underbrace{(\alpha_1 + \alpha_2 + \alpha_3)}_{tr(\underline{\underline{A}})} \underline{\underline{I}} + 2\mu \underbrace{(\alpha_1 \underline{\underline{P}}_{v_1} + \alpha_2 \underline{\underline{P}}_{v_2} + \alpha_3 \underline{\underline{P}}_{v_3})}_{\underline{\underline{A}}}$$

Representation for linear isotropic Tensor function

An linear isotropic function $\underline{\underline{G}}(\underline{\underline{A}})$ that maps symmetric tensors $\underline{\underline{E}}$ into symmetric tensors $\underline{\underline{G}}(\underline{\underline{A}})$ must have following form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \lambda\, tr(\underline{\underline{A}})\, \underline{\underline{I}} + 2\mu \underline{\underline{A}}} \qquad \underline{\underline{A}} = \underline{\underline{A}}^T$$

where $\lambda, \mu \in \mathbb{R}$ are scalars

In terms of a non-symetric tensor $\underline{\underline{E}} \neq \underline{\underline{E}}^T$

$$\boxed{\underline{\underline{G}}(\underline{\underline{E}}) = \lambda\, tr(\underline{\underline{E}})\, \underline{\underline{I}} + \mu\, sym(\underline{\underline{E}})}$$

$\Rightarrow$ standard constitutive laws

Linear elasticity: $\underline{\underline{E}} = \nabla \underline{u}$

Newtonian fluid: $\underline{\underline{E}} = \nabla \underline{v}$

Show λ & μ are independent of $\underline{v}_i$:

$$|\underline{e}| = |\underline{f}| = 1 \qquad \underline{\underline{R}}\,\underline{e} = \underline{f} \qquad \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}} \qquad \det(\underline{\underline{R}}) = 1$$

$$\Rightarrow \underline{\underline{P}}_f = \underline{f} \otimes \underline{f} = (\underline{\underline{R}}\,\underline{e}) \otimes (\underline{\underline{R}}\,\underline{e}) = \underline{\underline{R}}(\underline{e} \otimes \underline{e})\underline{\underline{R}}^T =$$

$$= \underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T$$

isotropic: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)\underline{\underline{R}}^T$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)\,\underline{\underline{R}}^T$$

$$\Rightarrow \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)\,\underline{\underline{R}}^T - \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \underline{\underline{0}}$$

subsituting: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \lambda(\underline{f})\underline{\underline{I}} + 2\mu(\underline{f})\,\underline{\underline{P}}_f$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e) = \lambda(\underline{e})\underline{\underline{I}} + 2\mu(\underline{e})\,\underline{\underline{P}}_e$$

$$[\lambda(\underline{e}) - \lambda(\underline{f})]\,\underline{\underline{I}} + 2[\mu(\underline{e}) - \mu(\underline{f})]\,\underline{\underline{P}}_f = \underline{\underline{0}}$$

since $\underline{\underline{I}}$ and $\underline{\underline{P}}_f$ are linearly independent

$$\Rightarrow \lambda(\underline{e}) = \lambda(\underline{f}) = \lambda \qquad \mu(\underline{e}) = \mu(\underline{f}) = \mu$$

λ & μ are constants

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i}) = \lambda\underline{\underline{I}} + 2\mu\,\underline{\underline{P}}_{v_i}$$