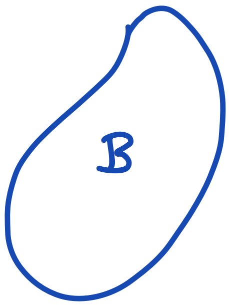
Continuum Mass and Force Concepts



Volume of a body B:

$$V_B = \int_B dV$$

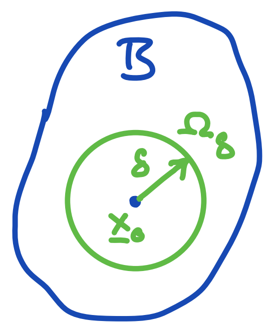
Mass of a body B:

$$m_B = \int_B \rho(\underline{x})\, dV$$

$\rho(\underline{x})$ = mass density field

At any point $\underline{x}_0$ in B

$$\boxed{\rho(\underline{x}_0) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}}$$



Important geometric quantities of a body are:

Center of volume: $\underline{x}_v = \dfrac{1}{V_B} \displaystyle\int_B \underline{x}\, dV$

Center of mass: $\underline{x}_m = \dfrac{1}{m_B} \displaystyle\int_B \rho(\underline{x})\, \underline{x}\, dV$

Note: $\rho = \text{const}$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho\, \underline{x}\, dV = \frac{\rho}{\rho V_\Omega} \int \underline{x}\, dV = \frac{1}{V_\Omega} \int_\Omega \underline{x}\, dV = \underline{x}_v$$

Important because resulting forces.

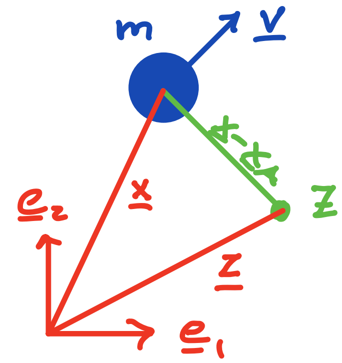
Short review of force and moment

Object with a mass m and velocity $\underline{v}$ has a momentum:

Linear momentum: $\underline{L} = m\,\underline{v}$

angular momentum: $\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$

$\rightarrow$ always relative to a point !



Newtons 1st law: "Principle of inertia"

In a fixed frame of reference every object preserves its state of motion unless it is acted upon by a force or torque.

Force: $\underline{f} = \frac{d\underline{L}}{dt} = \frac{d(m\underline{v})}{dt} = m\frac{d\underline{v}}{dt} = m\,\underline{a}$ $\qquad \frac{d\underline{v}}{dt} = \dot{\underline{v}}$

$\left[\frac{ML}{T^2}\right] = N$ $\qquad \boxed{\underline{f} = m\,\underline{a}}$ $\qquad$ Newton's 2nd law

Torque: $\underline{\tau} = \frac{d\underline{j}}{dt} = m\frac{d}{dt}(\underline{x}\times\underline{v} - \underline{z}\times\underline{v}) =$

(moment of force) $= m(\dot{\underline{x}}\times\underline{v} + \underline{x}\times\dot{\underline{v}} - \cancel{\dot{\underline{z}}}^{\,0}\times\underline{v} - \underline{z}\times\dot{\underline{v}})$

$\left[\frac{ML^2}{T^2}\right] = Nm$ $\qquad = m(\cancel{\underline{v}\times\underline{v}}^{\,0} + \underline{x}\times\underline{a} - \underline{z}\times\underline{a}) = m(\underline{x}-\underline{z})\times\underline{a}$

$\underline{\tau} = (\underline{x}-\underline{z})\times m\,\underline{a} = (\underline{x}-\underline{z})\times\underline{f}$

Body Forces

Any force that not due to physical contact is a body force and acts on the entire body. Common body forces originate from gravitational and electromagnetic fields.

Example: gravitational body force

$$\boxed{\underline{b}_g = \rho \underline{g}} \qquad \left[\frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}\right]$$

$\Rightarrow$ body force field has units of $\frac{\text{force}}{\text{volume}}$

If a body force acts on a body B the net or resultant body force is:

$$\boxed{\underline{r}_b[B] = \int_B \underline{b}(\underline{x})\, dV} \quad \text{units of force } \left[\frac{ML}{T^2}\right].$$

The net or resultant torque on a body about $\underline{z}$

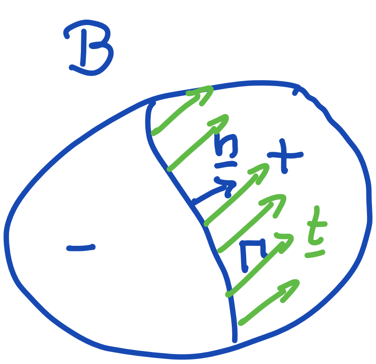
$$\boxed{\underline{\tau}_b[B] = \int_\Omega (\underline{x} - \underline{z}) \times \underline{b}(\underline{x})\, dV}$$

Surface/Contact Forces

arise due to the physical contact between bodies. Forces along imaginary surfaces within a body are called internal forces while forces along the bounding surface of a body are external.

Internal surface forces hold a body together. External surface forces describe the interaction with the environment.

Traction Field



Consider an arbitrary surface Γ in B with unit normal $\underline{n}(\underline{x})$ that defines the positive and negative sides of B.

The force per unit area exerted by material on the pos. side upon material on the neg. side is given by the traction field $\underline{t}_n$ for Γ.

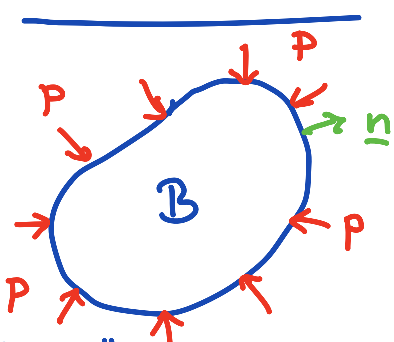
The resultant force due to a traction field on Γ is

$$\boxed{\underline{F}_s[\Gamma] = \int_\Gamma \underline{t}_n(\underline{x})\, dA}$$

The resultant torque about point $\underline{z}$ due to a traction field on Γ is

$$\boxed{\underline{T}_s[\Gamma] = \int_\Gamma (\underline{x}-\underline{z}) \times \underline{t}_n(\underline{x})\, dA}$$

Example: Pressure, p, on submerged body

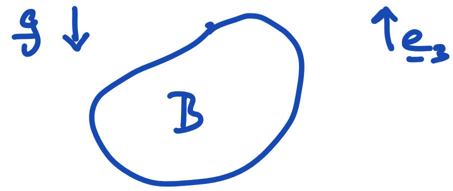


$$\boxed{\underline{t} = -p\,\underline{n}}$$

"Hydrostatic surface force"

Weight: Resultant gravitational body force

The weight of a body is the resultant force due to gravity.



$$\underline{f}_G = \underline{r}_b[\mathcal{B}] = \int_{\mathcal{B}} \rho_{\mathcal{B}}\, \underline{g}\, dV$$

if $\rho_{\mathcal{B}}$ = const. and g = const. $\qquad \underline{g} = -g\,\underline{e}_3$

$$\underline{f}_G = -\rho_{\mathcal{B}}\, g\, \underline{e}_3 \int_{\mathcal{B}} dV = -\rho_{\mathcal{B}}\, g\, V_{\mathcal{B}}\, \underline{e}_3 = -m_{\mathcal{B}}\, g\, \underline{e}_3 = m_{\mathcal{B}}\, \underline{g}$$

Acceleration of a free falling body in vacuum

$$\underline{f}_G = m_{\mathcal{B}}\, \underline{a} \quad \Rightarrow \quad \underline{a} = \frac{1}{m_{\mathcal{B}}}\, \underline{f}_G = -g\, \underline{e}_3$$

$\boxed{\underline{a} = -g\, \underline{e}_3}$ acceleration during free fall is independent of mass (Galileo)

Q: Where on $\mathcal{B}$ does $\underline{f}_G$ act?

Moment of Gravity

Resultant torque on body $\mathcal{B}$ about origin, $\underline{x} = \underline{0}$, due to gravitational body force:

$$\boxed{\underline{\tau}_G = \underline{\tau}_b[\mathcal{B}]_{\underline{0}} = \int_{\mathcal{B}} \underline{x} \times \rho(\underline{x})\, \underline{g}\, dV} \qquad \underline{g} = -g\,\underline{e}_3$$

Resultant torque around $\underline{x}_m$

$$\underline{T}_b[B] = \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} \, dV \qquad \text{note } \underline{x}_m = \text{const}$$

$$= \int_B \underline{x} \times \rho \underline{g} - \underline{x}_m \times \rho \underline{g} \, dV$$

$$= \underbrace{\int_B \underline{x} \rho \, dV}_{\underline{x}_m m_B} \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho \, dV}_{m_B}$$

$$= m_B \, \underline{x}_m \times \underline{g} - m_B \, \underline{x}_m \times \underline{g} = \underline{0}$$

$\Rightarrow$ gravitational torque around $\underline{x}_m$ vanishes

Simplify "moment of gravity"

$$\underline{T}_G = \int_B \underline{x} \times \rho \underline{g} \, dV = \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho \underline{g} \, dV$$

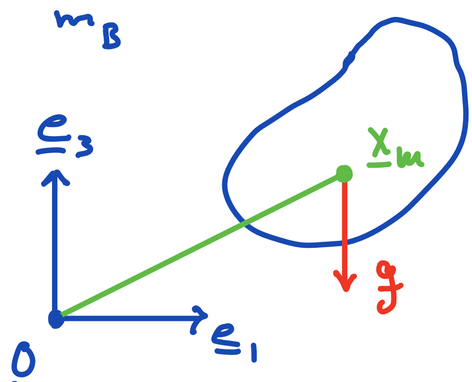
$$= \int_B \cancelto{\underline{0}}{(\underline{x} - \underline{x}_m)} \times \rho \underline{g} \, dV + \int_B \underline{x}_m \times \rho \underline{g} \, dV$$

$$= \int_B \underline{x}_m \times \rho \underline{g} \, dV = \underline{x}_m \times \underline{g} \underbrace{\int_B \rho \, dV}_{m_B}$$

$$\Rightarrow \quad \boxed{\underline{T}_G = \underline{x}_m \times m_B \, \underline{g}}$$

Moment of gravity

(torque about origin)



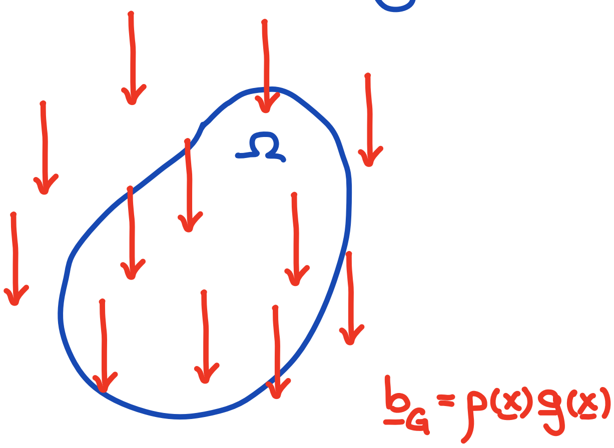
We say that: **Gravity acts on the center of mass.**

Because resultant torque about $\underline{x}_m$ is zero.

$\Rightarrow$ **Center of Mass Theorem** (prove it later)

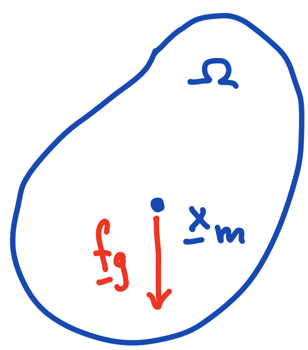
Provides the link between continuum & discrete!

Continuum system



gravitational body force field acts everywhere

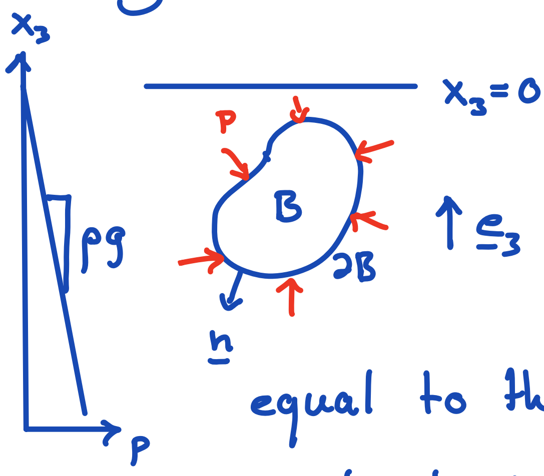
Discrete body



gravity vector $\underline{f}_g$ acts only on $\underline{x}_m$

$\Rightarrow$ force field can be represented as acting on the point where it does not induce a torque.

Buoyancy: Resultant hydrostatic surface force



Any object, wholly or partially submerged in a fluid is bouyed up by a force equal to the weight of the fluid displaced by the body (Archimedes principle).

Q: Is the buoyancy force a body or a surface force?

Hydrostatic pressure acts on the boundary of the object. $\Rightarrow$ external surface force

Buoyancy force $\rightarrow$ resultant surface force

$$\boxed{\underline{r}_s[\partial B] = \underline{f}_B = W\underline{e}_3 = \rho_f g V_B \underline{e}_3 = -m_f \underline{g}}$$

ρ_f = fluid density

Hydrostatic pressure: $p = -\rho_f g x_3$

Hydrostatic traction on ∂B: $\underline{t} = -p\underline{n}$

Resulting surface force:

$$\underline{F}_s[\partial B] = \int_{\partial B} \underline{t}\, dA = -\int_{\partial B} p\, \underline{n}\, dA$$

need to convert this to volume integral

$\Rightarrow$ Gradient theorem $\boxed{\int_{\partial\Omega} \phi\, \underline{n}\, dA = \int_{\Omega} \nabla \phi\, dV}$ → HW

$$\Rightarrow \quad \underline{F}_s[\partial B] = -\int_{\partial B} p\, \underline{n}\, dA = -\int_{B} \nabla p\, dV$$

where $\nabla p = \nabla(-\rho_f g x_3) = -\rho_f g\, \underline{e}_3$

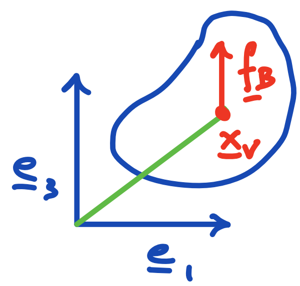
$$\underline{F}_s[\partial B] = \int_{B} \rho_f g\, \underline{e}_3\, dV = \rho_f g\, \underline{e}_3 \underbrace{\int_{B} dV}_{V_B} = \rho_f g V_B\, \underline{e}_3 \quad ✓$$

$$\underline{F}_s[\partial B] = m_f\, g\, \underline{e}_3 \quad = -\underline{F}_b[B] \text{ of fluid}$$

Moment of Buoyancy

With arguments similar to those used for gravity, we can show that buoyancy force has zero resultant torque about center of volume $\underline{x}_V = \frac{1}{V_B} \int_B \underline{x} \, dV$.

$$\boxed{\underline{\tau}_B = -\underline{x}_V \times (m_f \underline{g})} = \underline{x}_V \times m_f g \underline{e}_3$$



m_f = mass of displaced fluid

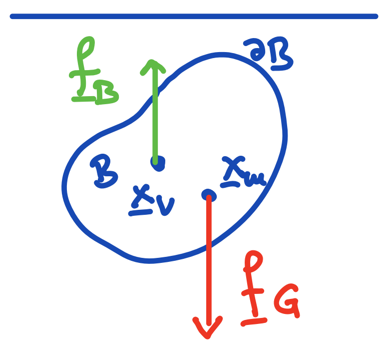
Buoyancy acts on center of volume.

$\Rightarrow$ center of buoyancy

The implicit assumption is that $\rho_f = \text{const.}$

otherwise the buoyancy froce acts on center of mass of displaced fluid (center of buoy.)

Hydrostatic force balance



Total resultant force $\underline{f}$ on a submerged body in a gravitational field is the sum of weight and buoyancy.

$$\underline{f} = \underline{f}_G + \underline{f}_B = \underline{r}_b[B] + \underline{r}_s[\partial B]$$

$$= -\int_B \rho_b g \underline{e}_3 \, dV - \oint_{\partial B} p \underline{n} \, dS$$

substituting:

$$\boxed{\underline{f} = \int_B (\rho_f - \rho_b) g \underline{e}_3 \, dV = (m_f - m_b) g \underline{e}_3}$$

$\rho_f > \rho_b$: $\underline{f}$ points up $\rightarrow$ body rises (pos. buoyancy)

$\rho_f < \rho_b$: $\underline{f}$ points down $\rightarrow$ body sinks (neg. buoyancy)

$\rho_f = \rho_b$: $\underline{f} = \underline{0}$ $\rightarrow$ body is neutrally buoyant

Note: The integrated expression assumes $g = \text{const.}$

Hydrostatic Moment of floating body

$\underline{f}_G$ and $\underline{f}_B$ act on different points

$\Rightarrow$ induce a moment / net torque

$$\underline{\tau}_G = \underline{x}_m \wedge m_B \underline{g} \quad \text{and} \quad \underline{\tau}_B = -\underline{x}_v \wedge m_f \underline{g}$$

For floating body: $\underline{f} = (m_b - m_f)\,\underline{g} = \underline{0}$

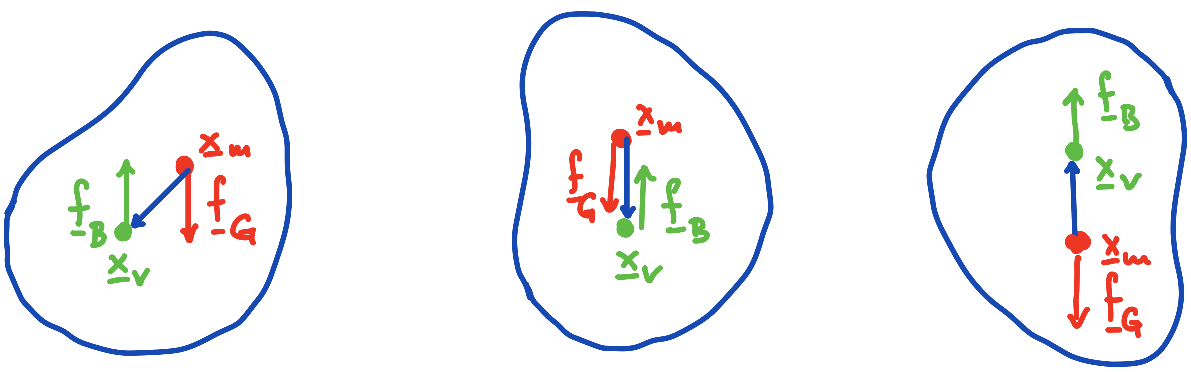
$$\Rightarrow m_b = m_f \equiv m$$

Total torque on body:

$$\underline{\tau} = \underline{\tau}_G + \underline{\tau}_B = \underline{x}_m \wedge m\underline{g} - \underline{x}_v \wedge m\underline{g}$$

$$\boxed{\underline{\tau} = (\underline{x}_m - \underline{x}_v) \wedge m\underline{g}}$$

Stability of fully submerged body:



$\underline{\tau} \neq \underline{0}$ unstable

$\underline{\tau} = \underline{0}$ meta stable

$\underline{\tau} = \underline{0}$ stable