Hydrostatic shape of Earth

Hydrostatic stress: $\underline{\underline{\sigma}} = -p\underline{\underline{I}}$ $\quad\quad$ $p = p_0 + \rho g$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = p_0 - \rho g \underline{x}_3$

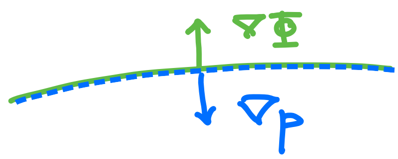
Substitute into eqbm

$$\nabla\cdot\underline{\underline{\sigma}} = \nabla\cdot(-p\underline{\underline{I}}) = -\nabla p$$

$$\Rightarrow \quad \nabla p = \rho \underline{g} \qquad\qquad \nabla p \parallel \underline{g}$$

Grav. field: $\underline{g} = -\nabla\Phi$ $\qquad$ Φ grav. potential

so that $\quad \boxed{\nabla p = -\nabla\Phi}$

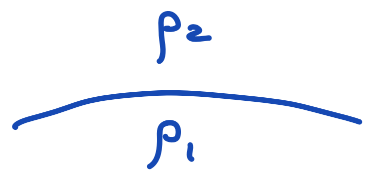


$\Longrightarrow$ Isobars and equipotential's are parallel

isobars: $\quad p(\underline{x}) = p_1$

equipotential: $\quad \Phi(\underline{x}) = \Phi_1$

Surface of planet is change in density of fluid



$\Rightarrow$ isopycnic

$$\underline{0} = \nabla\times\nabla p = \nabla\times(\rho\underline{g}) = \nabla\rho\times\underline{g} + \rho\nabla\times\underline{g}$$

$$= -\nabla\rho\times\nabla\Phi - \rho\,\cancelto{0}{\nabla\times\nabla\Phi}$$

$$\nabla p \times \nabla \Phi = 0 \quad \Rightarrow \quad \nabla p \parallel \nabla \Phi$$

$\Rightarrow$ surface of body in hydrostatic eqilibrium is an equipotential surface

I) Stationary body

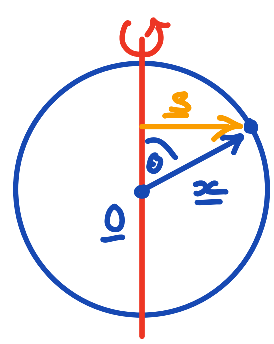
For a homogeneous spherical body for radius R:

$$\boxed{\Phi = \begin{cases} \frac{4}{6}\pi G\rho(r^2 - R^2), & r \leq R \\ -3MG\left(\frac{1}{r^2} - \frac{1}{R^2}\right), & r > R \end{cases}}$$

$\Rightarrow \Phi = \Phi(r)$ equipotentials are spheres

II) Rotating body

In frame rotating with body need to add centrifugal force $\underline{f}_c = m\Omega^2 s$



Ω = angular velocity

s = distance from axis

$\underline{s} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad s = |\underline{s}|$

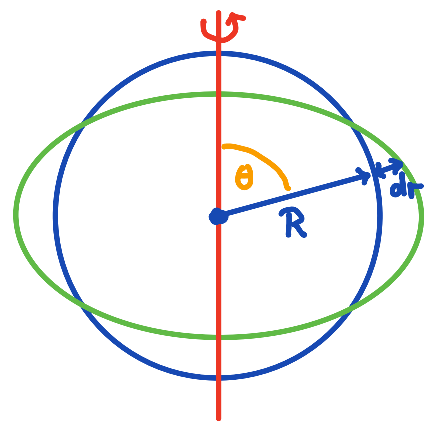
θ = polar angle $\qquad \underline{s} = \underline{x} \sin\theta$

adds a "fictitious" acceleration & potential

centrifugal: $g_c = \Omega^2 \underline{s}$ $\qquad \Phi_c = -\frac{1}{2}\Omega^2 s^2$

gravitational: $g_G = \frac{GM}{R^2}$ $\qquad \Phi_G \propto |g|\, dr = \frac{MG}{R^2} dr$

Note Φ_G is linearized at $r=R$ and $dr(\theta)$ is deviation from sphericity



Total potential:

$\Phi = \Phi_G + \Phi_c + \Delta\Phi$ $\qquad \Delta\Phi$ = self-potential due to deformation

ignoring $\Delta\Phi$

$\Phi = \frac{MG}{R^2} dr(\theta) - \frac{\Omega^2 R^2}{2} \sin^2\theta$

solving for dr

$dr = \frac{R}{2} q \sin^2\theta + \text{const}$ $\qquad q = \frac{\Omega^2 R}{|g|} = \frac{\Omega^2 R^3}{MG}$

Constant from mass/volume conservation

$$\int_0^{\pi} dr(\theta)\, dS = 2\pi R^2 \int_0^{\pi} dr(\theta) \sin\theta \, d\theta = 0$$

$$\Rightarrow \quad \boxed{dr = dr_0 \left(\sin^2\theta - \frac{2}{3}\right)} \qquad dr_0 = \frac{1}{2} R q$$

Earth: $R = 6371$ km $\quad |g| = 9.81 \frac{m}{s^2} \quad \Omega = 76 \cdot 10^{-6} \frac{1}{s}$

$$\Rightarrow q \approx 3.5 \cdot 10^{-3} \quad \Rightarrow dr_0 \approx 11 \text{ km}$$

actual $dr_0 = 21.4$ km

Error is due to self-potential

$$\boxed{\Delta\Phi = -\frac{3}{5} \frac{\rho_1}{\rho_0} g_s \, dr(\theta)}$$

where $\rho_0 = \frac{M}{V}$ and ρ_1 = density of shifted material

$$\Rightarrow \quad dr_0 = \frac{1}{2} R q \left(1 - \frac{3}{5} \frac{\rho_1}{\rho_0}\right)^{-1} \approx 21.9 \text{ km}$$

for $\rho_0 = 5500 \frac{kg}{m^3} \quad \rho_1 = 4500 \frac{kg}{m^3}$