Geo 325C Lecture 1:

Index notation:

1) Dummy indices

$\{ \underline{e}_1 \; \underline{e}_2 \; \underline{e}_3 \}$

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \boxed{\sum_{i=1}^{3} a_i \underline{e}_i \equiv a_i \underline{e}_i}$$

Einstein summation convention

if index is repeated twice → summation

$$\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k = a_\alpha \underline{e}_\alpha$$

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j)$$

$$= (a_n \underline{e}_n \cdot (b_j \underline{e}_j)$$

$$\neq (a_j \underline{e}_j) \cdot (b_j \underline{e}_j) \quad \text{not o.k.}$$

2) Free indices

occurs only once

Example: $\quad a_i = c_j \, b_j \, b_i$ $\qquad i$ = free index

j = dummy index

Short hand for set of equations: $i = 1, 2, 3$

$i=1$: $\quad a_1 = \left(\sum_{j=1}^{3} c_j b_j\right) b_1$

$i=2$: $\quad a_2 = \left(\sum_{j=1}^{3} c_j b_j\right) b_2$

$i=3$: $\quad a_3 = \left(\sum_{j=1}^{3} c_j b_j\right) b_3$

Basis: $\{ e_i \}$

Note:
- all terms must have same free indices
- there can be more than one free index
- cannot use same symbol for both dummy and free index
- dummy can only be repeated twice

Why are following expressions meaningless?

1) $a_i = b_j$ free index is not same

2) $a_i b_j = c_i d_j d_j$ 'j' is both dummy & free

3) $a_i b_j = c_i c_k d_k d_j + d_p c_l c_l d_q$

4) $a_i = b_k c_k d_k e_i$

To express standard vector operations ~~to~~ in index notation we need to introduce new symbols:

Kronecker delta

For any frame $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$

is result of orthonormal basis

Properties: 1) $\delta_{ij} = \delta_{ji}$ symmetry

2) $\underline{e}_i = \delta_{ij}\, \underline{e}_j$ transfer property

$$i=1: \quad \delta_{ij}\, \underline{e}_j = \delta_{11}\, \underline{e}_1 + \cancel{\delta_{12}}^{0}\, \underline{e}_2 + \cancel{\delta_{13}}^{0}\, \underline{e}_3 = \underline{e}_i$$

Example: Projection onto basis

$$\underline{u} \cdot \underline{e}_j = (u_i \underline{e}_i) \cdot \underline{e}_j = u_i (\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}}) = \delta_{ij}\, u_i = u_j$$

Example: Scalar product

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j (\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}})$$

$$= \delta_{ij}\, a_i\, b_j = a_i\, b_i =$$

$$= a_j\, b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Kronecker delta is dotproduct in index notation