Lecture 10: Mohr Circle

Logistics: - HW 1 will update resubmissions soon

- HW 2 now have 6/7 thank you
- HW 3 4/7 please submit
- HW 4 posted later today

Last time: Extremal Stress values

$\Rightarrow$ constrained optimization problem

- normal stresses $\Rightarrow \boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}}$

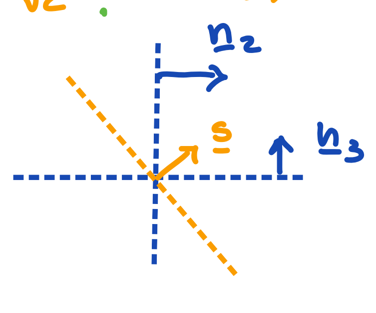
eigenvalues $\lambda_p = \sigma_p$ extremal normal stresses

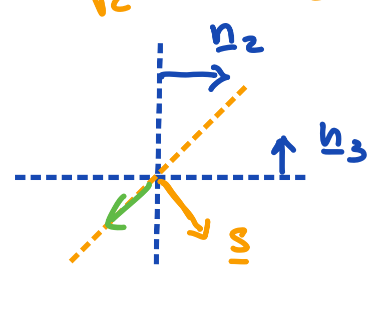
Note we assume: $\sigma_1 > \sigma_2 > \sigma_3 \geq 0$

$\underline{v}_p$ are normals to associated planes

- shear stresses $\boxed{\tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3)}$, $\underline{s} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3)$

$\underline{s} = \frac{1}{\sqrt{2}}(\underline{n}_2 + \underline{n}_3)$ $\qquad$ $\underline{s} = \frac{1}{\sqrt{2}}(\underline{n}_2 - \underline{n}_3)$





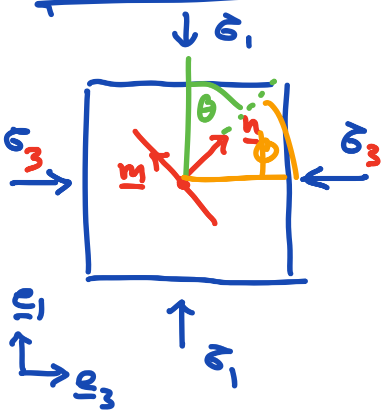
Today: - Mohr circle

- Computing stress on fault

Mohr circle

graphical way to display normal & shear stresses on all planes.

2D case



θ is angle between $\underline{n}\cdot\underline{e}_1 = \cos\theta$

ϕ is angle between $\underline{n}\cdot\underline{e}_3 = \cos\phi$

$\phi + \theta = \frac{\pi}{2}$ $\qquad \phi = \frac{\pi}{2} - \theta$

$\underline{n} = n_1\underline{e}_1 + n_3\underline{e}_3 \qquad n_2 = 0$

$n_1 = \underline{n}\cdot\underline{e}_1 = \cos\theta$

$n_3 = \underline{n}\cdot\underline{e}_3 = \cos\phi = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$$\Rightarrow \quad \underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \underline{m} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Stress in principal frame:

$$\underline{\underline{\sigma}} = \sigma_1\, \underline{e}_1 \otimes \underline{e}_1 + \sigma_2\, \underline{e}_2 \otimes \underline{e}_2 + \sigma_3\, \underline{e}_3 \otimes \underline{e}_3$$

traction: $\underline{t}_n = \underline{\underline{\sigma}}\, n = \sigma_1 \cos\theta\, \underline{e}_1 + \sigma_3 \sin\theta\, \underline{e}_3$

normal stress: $\sigma_n = \underline{n} \cdot \underline{t}_n = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$

use $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

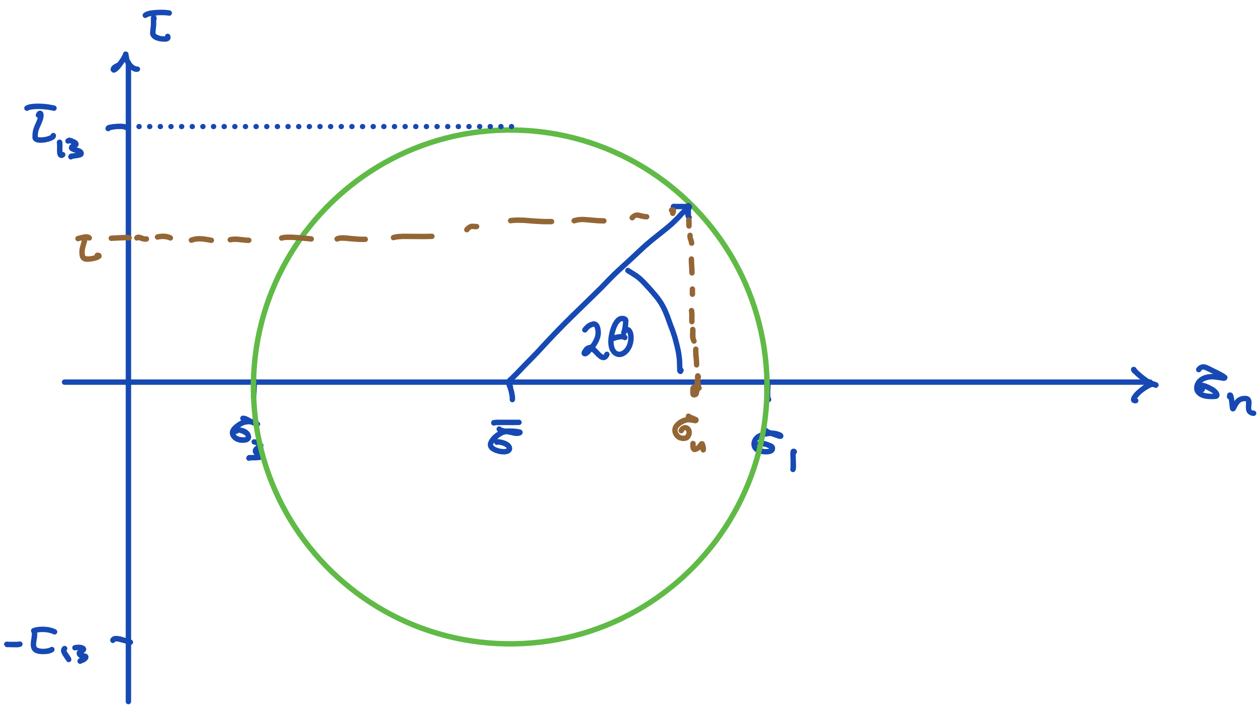
$$\Rightarrow \boxed{\sigma_n = \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2} \underline{\cos 2\theta}}$$

shear stress: $\tau = \underline{m} \cdot \underline{t}_n = (\sigma_1 - \sigma_3) \sin\theta \cos\theta$

use $2 \sin\theta \cos\theta = \sin(2\theta)$

$$\Rightarrow \boxed{\tau = \frac{\sigma_1 - \sigma_3}{2} \underline{\sin 2\theta}}$$

Together these equations for a circle in $\tau\sigma_n$- space with radius $R = \dfrac{\sigma_1 - \sigma_3}{2}$ and center $\Big(\underbrace{\dfrac{\sigma_1+\sigma_3}{2}}_{\bar{\sigma}}, 0\Big)$



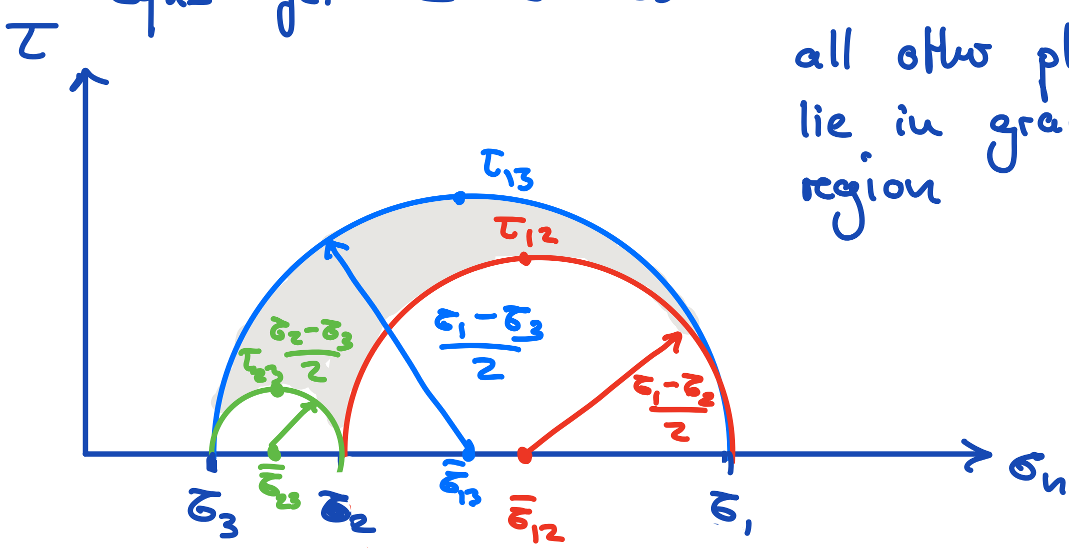
$\Rightarrow$ plane parallel to $\underline{e}_2$

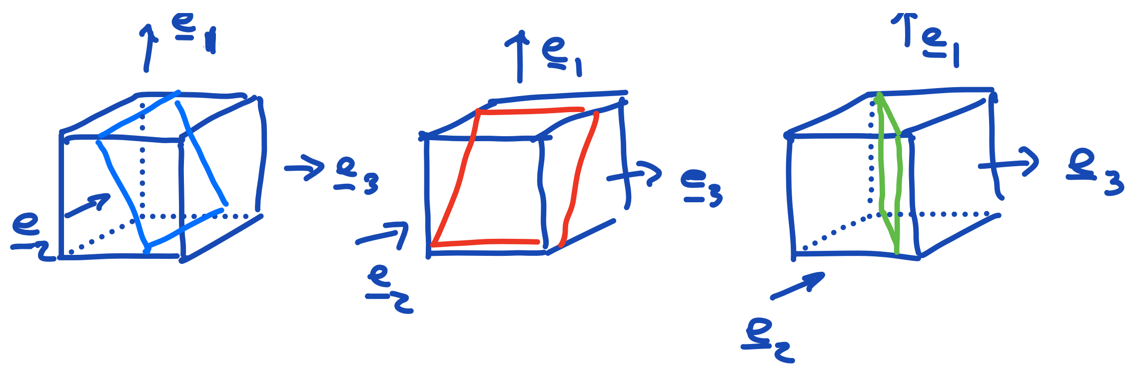
Mohr circles in 3D

Repeat argument for planes parallel to $\underline{e}_1$ & $\underline{e}_3$

$\Rightarrow$ eqns for 2 circles

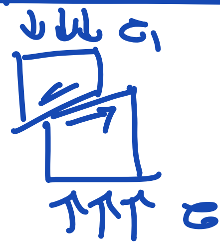
all other planes lie in gray shaded region





Failure criteria for shear fracture

most common type of brittle failure

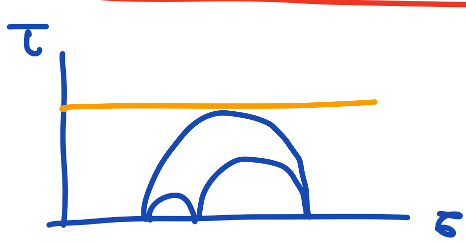


Empirical criteria for shear failure

I) Tresca criterion

Fracture occurs when max shear stress $\tau_{max} = \tau_{13}$ reaches the shear strength σ_Y

$$\boxed{|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \sigma_Y}$$



not affected by mean stress or intermediate stress

II Coulomb criterion

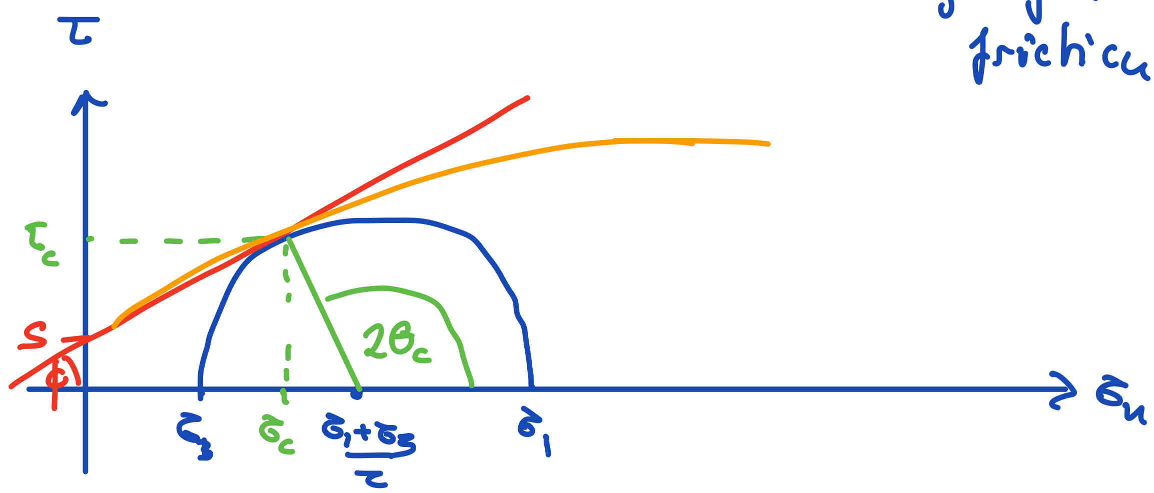
Fracture depends on both mag. of shear and normal stress

$$\boxed{|\tau| = S + \mu' \sigma_n}$$

S = cohesive strength

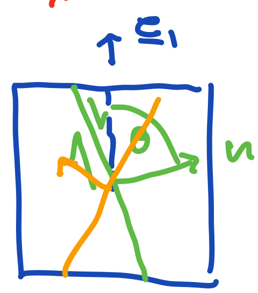
$\mu' = \tan\phi \approx 0.6$ internal fric

angle of internal friction $\phi \approx 30°$



failure occurs $\tau_c < \tau_{max}$

$\theta_c \sim 60°$



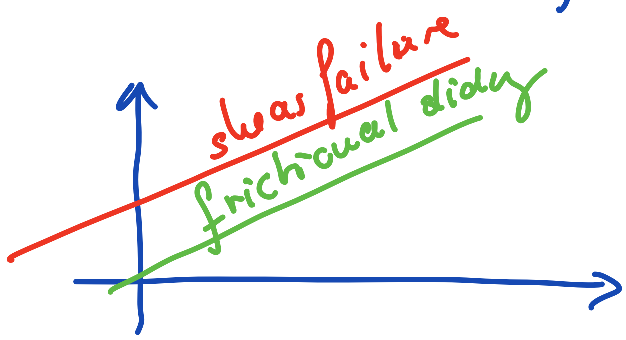
Byerlee's law

criterion for frictional sliding on fault

$$\boxed{|\tau| = S_0 + \mu_0 \sigma_n}$$

S_0 = cohesion of fault ~1-10 MPa

μ_0 = coefficient of friction ~ 0.5-0.8



Strength of brittle rocks is determined by frictional sliding