Lecture 12: Tensor Calculus, yey!

Logistics: - HW4 due today 3/7

- HW5 posted today

Last time: - Stress on a fault

- NED vs XYZ frames

strike dip

- Fault normals: $\boxed{\underline{n} = \underline{\underline{Q}}_s \underline{\underline{Q}}_D (-\underline{e}_3)}$

- Change of basis tensor $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$

$$\underline{\underline{A}} = ([\underline{e}_1'] \; [\underline{e}_2'] \; [\underline{e}_3'])$$

↑ representation of primed base in unprimed frame.

post Matlab example

- Stress tensor in principal frame.

→ get principal directions

- Representation of $\underline{\underline{\sigma}}$ in NED frame

- careful with order of eigenvalues!

Today: - Div, Grad, Curl and all that

Differentiation of Tensor Fields

A field is a function of space

scalar field: $\phi(\underline{x})$ temp., density

vector fields: $\underline{v}(\underline{x})$ force, velocity

tensor fields: $\underline{\underline{S}}(\underline{x})$ stress, therm. cond.

Review & extension of multivariable calculus.

$\Rightarrow$ Divergence, gradient, curl

Gradient

Gradient of a scalar field

$\phi(\underline{x})$ is differentiable at $\bar{\underline{x}}$ if there exists a vector field $\nabla\phi \in \mathcal{V}$ s.t.

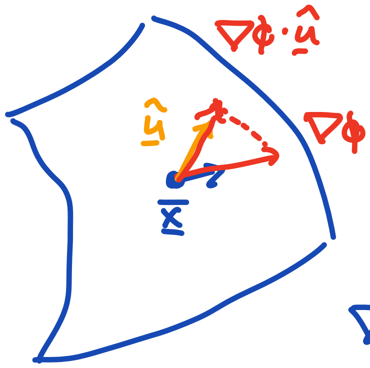
$$\boxed{\phi(\bar{\underline{x}}+\underline{h}) = \phi(\bar{\underline{x}}) + \nabla\phi(\bar{\underline{x}}) \cdot \underline{h} + \text{h.o.t}}$$

by Taylor expansion.

$$\boxed{\nabla\phi(\bar{\underline{x}}) \cdot \hat{\underline{u}} = \frac{d}{d\epsilon}\, \phi(\bar{\underline{x}}+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0}}$$

Directional derivative $\quad \underline{h} = \epsilon\,\hat{\underline{u}} \qquad |\hat{\underline{u}}| = 1$

$\nabla\phi$ is called gradient of ϕ



level set of $\phi(\underline{x}) = \phi_1$

$\nabla\phi \parallel \underline{n}$ in dir. of increasing ϕ

$$\underline{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$\nabla\phi$ is direction in which ϕ ~~changes~~ increases fastest.

Directional derivative at $\bar{\underline{x}}$ in dir $\underline{u}$

$$\nabla\phi(\underline{x})\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\phi(\underline{x}+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0} = D_{\hat{\underline{u}}}\phi(\underline{x})$$

Representation of $\nabla\phi$ in $\{\underline{e}_i\}$

$$\phi(\underbrace{\bar{\underline{x}}+\epsilon\hat{\underline{u}}}_{\underline{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}) = \phi(\underbrace{\bar{x}_1+\epsilon\hat{u}_1}_{x_1}, \underbrace{\bar{x}_2+\epsilon\hat{u}_2}_{x_2}, \underbrace{\bar{x}_3+\epsilon\hat{u}_3}_{x_3})$$

$$\nabla\phi(\underline{x})\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\phi(\bar{x}_1+\epsilon\hat{u}_1, \bar{x}_2+\epsilon\hat{u}_2, \bar{x}_3+\epsilon\hat{u}_3)\Big|_{\epsilon=0}$$

$$= \frac{\partial\phi}{\partial x_1}\frac{dx_1}{d\epsilon} + \frac{\partial\phi}{\partial x_2}\frac{dx_2}{d\epsilon} + \frac{\partial\phi}{\partial x_3}\frac{dx_3}{d\epsilon}\Big|_{\epsilon=0}$$

$$= \frac{\partial\phi}{\partial x_1}\hat{u}_1 + \frac{\partial\phi}{\partial x_2}\hat{u}_2 + \frac{\partial\phi}{\partial x_3}\hat{u}_3\Big|_{\epsilon=0}$$

$$= \phi_{,i}\hat{u}_i = \phi_{,i}\hat{u}_j\delta_{ij} = \phi_{,i}\hat{u}_j(\underline{e}_i\cdot\underline{e}_j)$$

Index notation for derivatives: $\boxed{\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}}$

$$\nabla \phi(\underline{x}) \cdot \underline{\hat{u}} = \underbrace{(\phi_{,i}\, \underline{e}_i)}_{\nabla\phi} \cdot \underbrace{(\hat{u}_j\, \underline{e}_j)}_{\underline{\hat{u}}}$$

Gradient: $[\nabla \phi] = \phi_{,i}\, \underline{e}_i = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{bmatrix}$

Gradient of a vector field

$\underline{v}(x)$ is differentiable at $\underline{x}$ if there exists a tensor field $\nabla \underline{v}(\underline{x}) \in \mathcal{V}^2$ s.t.

$$\boxed{\underline{v}(\bar{\underline{x}} + \underline{h}) = \underline{v}(\bar{\underline{x}}) + \nabla v(\bar{\underline{x}})\, \underline{h} + \text{h.o.t}}$$

where

$$\boxed{\nabla \underline{v}\, \underline{\hat{u}} = \frac{d}{d\epsilon} \underline{v}(\bar{\underline{x}} + \epsilon \underline{\hat{u}}) \Big|_{\epsilon=0}}$$ for all $\underline{u} \in \mathcal{V}$

$\underline{h} = \epsilon\, \underline{\hat{u}}$

In $\{\underline{e}_i\}$ $\quad \underline{v} = v_i\, \underline{e}_i \quad v_i = v_i(x_1, x_2, x_3)$

$\underline{\hat{u}} = u_k\, \underline{e}_k \quad \bar{\underline{x}} = \bar{x}_j\, \underline{e}_j$

i-th component

$$v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = v_i(\underbrace{\bar{x}_1 + \epsilon \hat{u}_1}_{x_1}, \bar{x}_2 + \epsilon \hat{u}_2, \bar{x}_3 + \epsilon \hat{u}_3)$$

by chain rule

$$\frac{d}{d\epsilon} v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = \frac{\partial v_i}{\partial x_1} \hat{u}_1 + \frac{\partial v_i}{\partial x_2} \hat{u}_2 + \frac{\partial v_i}{\partial x_3} \hat{u}_3 = \frac{\partial v_i}{\partial x_j} \hat{u}_j$$

For full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \hat{\underline{u}} = \frac{d}{d\epsilon} \underline{v}(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left(v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\underline{e}_i\right)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\left(v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\right)\Big|_{\epsilon=0} \underline{e}_i = \frac{\partial v_i}{\partial x_j} \hat{u}_j \underline{e}_i$$

$$= v_{i,j} \hat{u}_j \underline{e}_i$$

$$A_{ij} u_j \underline{e}_i = \underline{\underline{A}}\, \underline{u}$$

Representation of $\nabla \underline{v}$ in $\{\underline{e}_i\}$

$$\boxed{\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j}$$

Explicit:

$$\nabla \underline{v} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_2^T \end{bmatrix}$$

Divergence of a vector field

Def: to any $\underline{v}(x)$ we associate a scalar field

$$\boxed{\nabla \cdot \underline{v} = tr(\nabla \underline{v})}$$

Representation in $\{\underline{e}_i\}$ $\underline{v}(\underline{x}) = \underline{v}_i(x)\,\underline{e}_i$

$$\boxed{\nabla \cdot \underline{v} = tr(\nabla v) = v_{i,i}}$$

If $\nabla \cdot \underline{v} = 0$ $\Rightarrow$ solenoidal or divergence free

If $\underline{v}$ is a displacement or velocity then

$\nabla \cdot \underline{v}$ is related to volume change or its rate.

Divergence of a tensor field

To any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ we associate a vector field

$\nabla \cdot \underline{\underline{S}} \in \mathcal{V}$ called divergence of $\underline{\underline{S}}$

$$\boxed{(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})} \quad \text{for all } \underline{a} \in \mathcal{V}$$

$\underline{a}$ is arbitrary but not field

In $\{\underline{e}_i\}$ $\quad \underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{a} = a_k\, \underline{e}_k$

$$\underline{q} = \underline{\underline{S}}^T \underline{a} \qquad q_j = S_{ij}\, a_i \qquad (q_i = S_{ji}\, a_j)$$

substitute

$$(\nabla\cdot \underline{\underline{S}})\,\underline{a} = \nabla\cdot(\underline{\underline{S}}^T\underline{a}) = \nabla\cdot \underline{q} = tr(\nabla \underline{q}) = q_{j,j}$$

$$= S_{ij,j}\, a_i = (S_{ij,j}\, \underline{e}_i)\cdot(a_k \underline{e}_k)$$

by arbitrariness of $\underline{a}$

$$\boxed{\nabla\cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence product rules

$\phi(x)\in\mathbb{R}$ $\underline{v}(\underline{x})\in\mathcal{V}$ $\underline{\underline{S}}(\underline{x})\in\mathcal{V}^2$

$$\boxed{\begin{aligned}
\nabla\cdot(\phi\,\underline{v}) &= \underline{v}\cdot\nabla\phi + \phi\,\nabla\cdot\underline{v}\\
\nabla\cdot(\phi\,\underline{\underline{S}}) &= \underline{\underline{S}}\,\nabla\phi + \phi\,\nabla\cdot\underline{\underline{S}}\\
\nabla\cdot(\underline{\underline{S}}^T\underline{v}) &= (\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}}:\nabla\underline{v}\\
\nabla(\phi\,\underline{v}) &= \underline{v}\otimes\nabla\phi + \phi\,\nabla\underline{v}
\end{aligned}}$$

Example: $\nabla\cdot(\underline{\underline{S}}^T\underline{v})$ $\qquad q = \underline{\underline{S}}^T\underline{v}$ $\qquad q_j = S_{ij}\, v_i$

$$\nabla\cdot \underline{q} = tr(\nabla \underline{q}) = q_{j,j} = (S_{ij}\, v_i)_{,j}$$

$$= S_{ij,j}\, v_i + S_{ij}\, v_{i,j}$$

$$\underline{\underline{A}}:\underline{\underline{B}} = A_{ij}B_{ij}$$

$$= S_{ij,j}\, v_k\, \delta_{ik} + \underline{\underline{S}}:\nabla\underline{v}$$

$$= S_{ij,j}\, v_k\, \underline{e}_i \cdot \underline{e}_k + \underline{\underline{S}} : \nabla \underline{v}$$

$$= (S_{ij,j}\, \underline{e}_i) \cdot (v_k\, \underline{e}_k) + \underline{\underline{S}} : \nabla \underline{v}$$

$$= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v}$$

Next time

$\nabla \times \underline{v} \rightarrow \epsilon_{ijk}$

$\nabla \cdot \nabla \underline{v} = \Delta \underline{v}$

Eqbm eqns