Lecture 13: Equilibrium Equations

Logistics: HW 3 is graded ✓

HW 5 is due Th

$\underline{v} \cdot \underline{u}$

Feed back on HW 3: $a \cdot \underbrace{Q^T Q}_{I} b$ $\qquad \underline{v}^T \underline{u}$

$(Qa)^T = a Q^T$

$\boxed{\underline{\underline{Q}}\,\underline{a} \cdot \underline{\underline{Q}}\,\underline{b} = \underline{a} \cdot \underline{b}}$ use $\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}$

$\underline{\underline{Q}}\,\underline{a} \cdot \underline{v} = \underline{a} \cdot \underline{\underline{Q}}^T \underline{v} = \underline{a} \cdot \underline{\underline{Q}}^T \underline{\underline{Q}}\,\underline{b} = \underline{a} \cdot \underline{\underline{I}}\,\underline{b} = \underline{a} \cdot \underline{b}$

Careful with indices in matrix vector product

$\underline{\underline{Q}}\,\underline{a} \stackrel{?}{=} Q_{mk} a_k \underline{e}_l$

$\underline{\underline{Q}}\,\underline{b} \stackrel{?}{=} Q_{gc} b_c \underline{e}_g$

Fourier law: $q = -\kappa \nabla T$

Darcy's law: $q = -\underline{\underline{K}} \nabla h$

Ohm's law: $q = -\frac{1}{R} \nabla \phi$

Last time: - Tensor calculus

- Gradient: $\nabla \phi = \phi_{,i} \underline{e}_i$ $\qquad \phi_{,i} = \frac{\partial \phi}{\partial x_i}$

$\boxed{\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j}$

- Divergence: $\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v}) = v_{i,i}$

$\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i$

Today: - Curl

- Integral theorems

- Equilibrium equations

Curl of a vector field

$\underline{v} \in \mathcal{V}$ we associate another vector field $\nabla \times \underline{v} \in \mathcal{V}$ defined by

$$\boxed{(\nabla \times \underline{v}) \times \underline{a} = (\nabla \underline{v} - \nabla \underline{v}^T)\,\underline{a}}$$ for all $\underline{a} \in \mathcal{V}$

$$\underline{\underline{T}} = \nabla \underline{v} - \nabla \underline{v}^T = 2\,\text{skew}(\nabla \underline{v}) = \begin{bmatrix} 0 & -T_1 & T_2 \\ T_1 & 0 & -T_3 \\ -T_2 & T_3 & 0 \end{bmatrix}$$

$\underline{w} = \nabla \times \omega$ is the axial vector of $\underline{\underline{T}}$

$\Rightarrow$ swells of rotation.

$\underline{r} \times \underline{u} = \underline{\omega} = \underline{\underline{R}}\,\underline{u} \qquad R_{ik} = \epsilon_{kji} r_i$

In index notation

$$w_j = \tfrac{1}{2} \epsilon_{ijk} T_{ik} = \tfrac{1}{2} \epsilon_{ijk} (v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}) \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad \text{flip } i \leftrightarrow k$$

$$w_j = \epsilon_{ijk} v_{i,k}$$

$$\boxed{\underline{\omega} = \nabla \times \underline{v} = \epsilon_{ijk}\, v_{i,k}\, \underline{e}_j}$$

cross product: $\underline{a} \times \underline{b} = \epsilon_{ijk}\, a_i\, b_j\, \underline{e}_k$

$$\nabla \times \underline{v} = \epsilon_{ijk}\, v_{i,j}\, \underline{e}_k$$

Explicitly: $\nabla \times \underline{v} = (v_{3,2} - v_{2,3})\,\underline{e}_1 + (v_{1,3} - v_{3,1})\,\underline{e}_2$
$+ (v_{2,1} - v_{1,2})\,\underline{e}_3$

Physical interpretation:
If $\underline{v}$ is velocity field then $\nabla \times \underline{v}$ measures the angular velocity.

If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}$ is irrotational/conservative

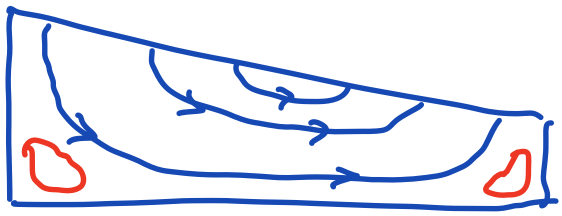
We can show:

$\boxed{\nabla \times \nabla \phi = 0}$ and $\boxed{\nabla \cdot (\nabla \times \underline{v}) = 0}$

Example: Darcy's law $\underline{q} = -K \nabla h$

$$\nabla \times \underline{q} = -K\, \nabla \times \nabla h = 0$$

$\Rightarrow$ singlephase groundwater flow is irrotational



Laplacian

To any scalar field $\phi \in \mathbb{R}$ we associate another scalar field $\Delta\phi = \nabla^2\phi$ defined by

$$\boxed{\Delta\phi = \nabla^2\phi = \nabla\cdot\nabla\phi}$$

$\{\underline{e}_i\}$ $\qquad \nabla\phi = \phi_{,i}\,\underline{e}_i$

$$\nabla\cdot\nabla\phi = tr(\nabla\nabla\phi) = tr(\phi_{,ij}\,\underline{e}_i\otimes\underline{e}_j) = \phi_{,ii}$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\boxed{\nabla^2\phi = \phi_{,ii}}$$ scaler Laplacian

governs : gravitational field, steady heat flow
inviscid flow ....

Vector Laplacian

To any $\underline{v} \in \mathcal{V}$ we associate another $\Delta \underline{v} \in \mathcal{V}$ defined by

$$\boxed{\Delta \underline{v} = \nabla^2 \underline{v} = \nabla \cdot \nabla \underline{v}}$$

$\{\underline{e}_i\}$ $\quad \underline{v} = v_i \, \underline{e}_i \quad \nabla \underline{v} = v_{i,j} \, \underline{e}_i \otimes \underline{e}_j$

$$\nabla \cdot \underline{\underline{S}} = S_{ij,j} \, \underline{e}_i$$

$$\boxed{\Delta \underline{v} = v_{i,jj} \, \underline{e}_i}$$ vector laplacian

$\Rightarrow$ governs viscous flow & static elasticity

Use fundamental identity

$$\boxed{\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v})}$$

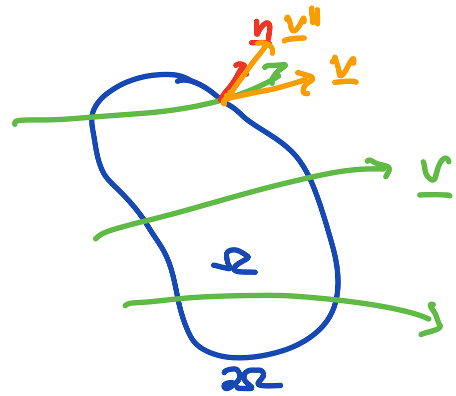
often used to simplify equation if $\nabla \cdot \underline{v} = 0$ or $\nabla \times \underline{v} = \underline{0}$ or both.

Integral laws

Essential to derive balance laws

Vector divergence Theorem

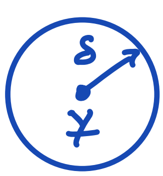
$$\boxed{\begin{gathered}\int_{\partial\Omega} \underline{v}\cdot\underline{n}\, dA = \int_{\Omega} \nabla\cdot\underline{v}\, dV \\ \int_{\partial\Omega} v_i\, n_i\, dA = \int_{\Omega} v_{i,i}\, dV\end{gathered}}$$



Physical interpretation

If $\underline{v}$ is velocity $\left[\frac{L}{T}\right]$ or volumetric flux $\left[\frac{L^3}{L^2 T} = \frac{L}{T}\right]$. then units $\int \underline{v}\cdot\underline{n}\, \underbrace{dA}_{L^2}$ are $\left[\frac{L^3}{T}\right]$ so that l.h.s represents rate at which volume is leaving or entering Ω.

Sphere



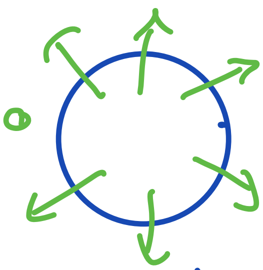
$$\int_{\partial\Omega_\delta} \underline{v}\cdot\underline{n}\, dA = \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV$$

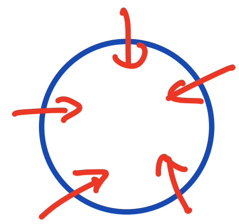
$$\lim_{\delta\to 0} \int_{\Omega} \nabla\cdot\underline{v}\, dV \approx V_\delta\, \nabla\cdot v\big|_{\underline{x}} \qquad V_\delta = \text{vol. sphere}$$

$$\Rightarrow \quad \boxed{\nabla \cdot \underline{v}\big|_x = \lim_{\delta \to 0} \frac{1}{V_\delta} \int_{\partial\Omega} \underline{v} \cdot \underline{n} \, dA}$$

Divergence is point wise rate of volume expansion or contraction

$\nabla \cdot \underline{v} > 0$





$\nabla \cdot v < 0$

Incompressible flows & deformations are solenoidal $\quad \nabla \cdot \underline{v} = 0$

Tensor divergence theorem

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{\underline{S}} \, \underline{n} \, dA &= \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, dV \\ \int_{\partial\Omega} S_{ij} n_j \, dA &= \int_{\Omega} S_{ij,j} \, dV \end{aligned}}$$

Can be derived from vector version

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}} \, \underline{n} \, dA = \int_{\partial\Omega} \underline{a} \cdot \underline{\underline{S}} n \, dA = \int_{\partial\Omega} \underline{\underline{S}}^T \underline{a} \cdot \underline{n} \, dA$$

$\underline{\underline{S}}^T \underline{a}$ is a vector so we can apply div. thm

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \underline{n} \, dA = \int_{\Omega} \nabla \cdot (\underline{\underline{S}}^T \underline{a}) \, dV$$

using definition: $(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})$

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \underline{n} = \int_{\Omega} (\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} \, dV \qquad \underline{a} \text{ is arbitrary}$$

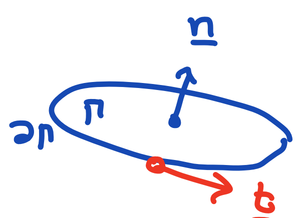
$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}} \, \underline{n} \, dA = \underline{a} \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, dV$$

$$\int_{\partial\Omega} \underline{\underline{S}} \, \underline{n} \, dA = \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, dV$$

Stokes Thm

surface Γ

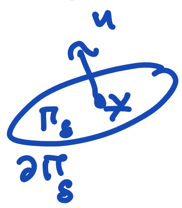
Φ



$$\boxed{\int_{\Gamma} (\nabla \times \underline{v}) \cdot \underline{n} \, dA = \oint_{\partial\Gamma} \underline{v} \cdot \underline{t} \, ds}$$

$\oint_{\partial\Gamma} \underline{v} \cdot \underline{t} \, ds$ is circulation of $\underline{v}$ around $\partial\Gamma$

Physical interpretation



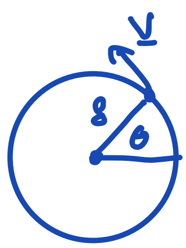
shrinking disk Γ_δ

limit $\delta \to 0$

$$\lim_{\delta\to 0} \oint_{\partial\Gamma} \underline{v}\cdot\underline{t}\, ds = \overline{\underline{v}\cdot\underline{t}}\Big|_{\underline{x}} 2\pi\delta$$

$$\lim_{\delta\to 0} \int_{\Gamma} (\nabla\times\underline{v})\cdot\underline{n}\, dA = \nabla\times\underline{v}\big|_{\underline{x}}\cdot\underline{n}\ \pi\delta^2$$

$$\Rightarrow \quad \overline{\underline{v}\cdot\underline{t}}\Big|_{\underline{x}} 2\pi\delta \approx \nabla\times\underline{v}\big|_{\underline{x}}\cdot\underline{n}\ \pi\delta^2$$



angular velocity: $\omega = \frac{d\theta}{dt}$

$|\underline{v}| = \omega\delta$

$$\Rightarrow \quad \overline{\underline{v}\cdot\underline{t}}\Big|_{y} = \omega\delta$$

substitute

$$2\cancel{\pi}\,\cancel{\delta^2}\ \omega \approx \nabla\times\underline{v}\big|_{\underline{x}}\cdot\underline{n}\ \cancel{\pi}\,\cancel{\delta^2}$$

$$2\omega = \nabla\times\underline{v}\big|_{\underline{x}}\cdot\underline{n} \qquad \underline{n} = \frac{\nabla\times v}{|\nabla\times\underline{v}|}\Big|_{\underline{x}}$$

$$= \frac{(\nabla\times\underline{v}|_{\underline{x}})\cdot(\nabla\times\underline{v}|_{y})}{|\nabla\times v|_y|} = |\nabla\times\underline{v}|_{\underline{x}}$$

$$\Rightarrow \quad \boxed{|\nabla\times\underline{v}|_{\underline{x}} = 2\omega}$$

Curl of $\underline{v}$ is twice the angular velocity

Example: Poissons equation for gravitational potential

grav. field: $\quad g = -\nabla \Phi$

field (vector) $\quad$ potential (scalar)

Gauss law of gravity:

$$\nabla \cdot g = -4\pi \rho G$$

grav. const.

Substituting:

$$\nabla \cdot (-\nabla \Phi) = -4\pi \rho G$$

$$\boxed{\Delta \Phi = 4\pi \rho G}$$ Poisson's Eqn