Lecture 14: Equilibrium Equations

Logistics: - HW5 due 3/7 ???

- HW6 will be posted

Last time: - curl $(\nabla \times \underline{v}) \times \underline{a} = (\nabla \underline{v} - \nabla \underline{v}^T)\, a$

$$\underline{\omega} = \nabla \times \underline{v} = \epsilon_{ijk}\, v_{i,k}\, \underline{e}_j$$

$$\nabla \times \nabla \phi = 0 \qquad \nabla \cdot (\nabla \times \underline{v}) = 0$$

- Laplacian : $\nabla \cdot \nabla = \Delta = \nabla^2$

scalar $\Delta \phi = \phi_{,ii}$

vector $\Delta \underline{v} = v_{i,jj}\, \underline{e}_i$

$$\boxed{\Delta \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times \nabla \times \underline{v}}$$

- Integral laws

Div. Theorem: $\int_{\partial\Omega} \underline{v} \cdot \underline{n}\, dS = \int_{\Omega} \nabla \cdot \underline{v}\, dV$

$$\int_{\partial\Omega} \underline{\underline{S}} \cdot \underline{n}\, dS = \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV$$

Stokes Thm: $\int_{\Gamma} (\nabla \times \underline{v}) \cdot \underline{n}\, dA = \oint_{\partial\Gamma} \underline{v} \cdot \underline{t}\, ds$

$$|\nabla \times \underline{v}| = 2\omega$$

Today: - Equilibrium equations

- Hydrostatic shapes

Mechanical Equilibrium

Neccessary condition for eqbm

$$\boxed{\begin{aligned} &\underline{F}[\Omega] = \underline{F}_b[\Omega] + \underline{F}_s[\partial\Omega] = \int_\Omega \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{t}\, dA = 0 \\ &\underline{T}[\Omega] = \underline{T}_b[\Omega] + \underline{T}_s[\partial\Omega] = \int_\Omega \underline{x} \times \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA = 0 \end{aligned}}$$

If $\underline{F}[\Omega] = 0$ then $\underline{T}[\Omega]$ is independent of $\underline{z}$!

Substitute definition of Cauchy stress $\underline{t} = \underline{\underline{\sigma}}\, \underline{n}$

$\underline{n}$ = unit normal (outward)

$$\underline{F}[\Omega] = \int_{\partial\Omega} \underline{\underline{\sigma}}\, \underline{n}\, dA + \int \rho \underline{b}\, dV = 0$$

Tensor divergence Them

$$\underline{F}[\Omega] = \int_\Omega \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}\, dV = 0$$

by arbitrariness of Ω we have

$$\underline{\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0}$$

$$\underline{T}[\Omega] = \int_{\partial\Omega} \underline{x} \times (\underline{\underline{\sigma}}\, \underline{n})\, dA + \int_\Omega \underline{x} \times \rho \underline{b}\, dV$$

subst: $\rho \underline{b} = -\nabla \cdot \underline{\underline{\sigma}}$

$$\underline{I}[\Omega] = \int_{\partial\Omega} \underline{x} \times (\underline{\underline{\sigma}}\,\underline{n})\, dA - \int_{\Omega} \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}})\, dV = 0$$

need to app. div. thm to $\partial\Omega \to \Omega$

$$\underline{x} \times (\underline{\underline{\sigma}}\, n) = \underline{\underline{R}}\, \underline{n} \qquad R_{iL} = \epsilon_{ijk}\, x_j\, \sigma_{kL}$$

$$\underline{I}[\Omega] = \int_{\partial\Omega} \underline{\underline{R}}\, \underline{n}\, dA - \int_{\Omega} \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}})\, dV$$

$$= \int_{\Omega} \nabla \cdot \underline{\underline{R}} - \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}})\, dV = 0$$

by arbitraryness of Ω

$$\nabla \cdot \underline{\underline{R}} - \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}}) = 0$$

in ~~coord~~ index notation

$$(\epsilon_{ijk}\, x_j\, \sigma_{kL})_{,L} - \epsilon_{ijk}\, x_j\, \sigma_{kl,L} = 0$$

chain rule

$$\epsilon_{ijk}\, x_{j,L}\, \sigma_{kL} + \cancel{\epsilon_{ijk}\, x_j}\, \sigma_{kl,L} - \cancel{\epsilon_{ijk}\, x_j}\, \sigma_{kl,t} = 0$$

$$\Rightarrow \qquad \epsilon_{ijk}\, x_{j,L}\, \sigma_{kL} = 0 \qquad\qquad x_{j,L} = \delta_{jk}$$

$$\epsilon_{ijk}\,\delta_{jk}\,\sigma_{kl} = \boxed{\epsilon_{ijk}\,\sigma_{kj} = 0}$$

If $\epsilon_{ijk}\sigma_{kj} = 0$ then $\epsilon_{ikj}\,\sigma_{jk} = 0$
because j, k are dummy indices

$$0 = \epsilon_{ijk}\,\sigma_{kj} + \epsilon_{ikj}\,\sigma_{jk} = \epsilon_{ijk}\left(\sigma_{kj} - \sigma_{jk}\right) = 0$$

Can choose i different from j & k $\epsilon_{ijk} \neq 0$

$$\Rightarrow \quad \boxed{\sigma_{kj} = \sigma_{jk}}$$

$\Rightarrow$ stress tensor is symmetric $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Equations for mechanical Eqbm

$$\boxed{\begin{array}{lll} \nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b} = 0 & \sigma_{ij,j} + \rho\, b_i \\ \underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} & \sigma_{ij} = \sigma_{ji} \end{array}}$$

force bal.

torque bal.

Hydrostatic shapes

$\nabla p =$

$p = p_0 + \rho g x_3$

$= p_0 - \rho g \underline{e}_3 x_3$

Hydrostatic stress: $\underline{\underline{\sigma}} = -p \underline{\underline{I}}$

subst. into eqbm eqns.

$$\nabla \cdot \underline{\underline{\sigma}} = \nabla \cdot (-p \underline{\underline{I}}) = -\underline{\underline{I}} \nabla p - p \cancelto{0}{\nabla \cdot \underline{\underline{I}}}$$

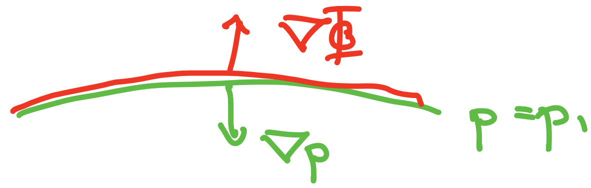
$$= -\nabla p$$

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g} = 0$$

$$\Rightarrow \quad \nabla p = \rho \underline{g} \quad \Rightarrow \quad \nabla p \parallel \underline{g}$$

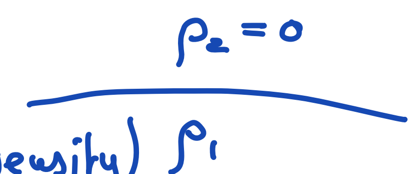
grav. acc.: $\underline{g} = -\nabla \Phi$

$$\boxed{\nabla p = -\nabla \Phi}$$



Isobars and equipotential surf. are parallel

Surface is change in ρ



isopycnic (surf of const density)

$$\nabla \times \nabla p = 0 \qquad \underline{g} = -\nabla \Phi$$

$$\nabla \times \nabla p = \nabla \times (\rho \underline{g}) = \nabla \rho \times \underline{g} + \rho \cancelto{0}{\nabla \times \underline{g}}$$

$$= -\nabla \rho \times \nabla \Phi = 0$$

$$\Rightarrow \quad \nabla \rho \parallel \nabla \Phi$$

I) Stationary body

Homogeneous body

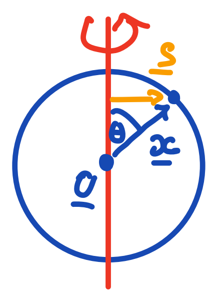
$$\boxed{\Phi = \begin{cases} \frac{4}{6}\pi G\rho\,(r^2 - R^2) & r \le R \\ -\frac{1}{3} MG \left(\frac{1}{r^3} - \frac{1}{R^3}\right) & r \ge R \end{cases}}$$

$\Phi = \Phi(r) \Rightarrow$ equipotentials are spheres

II Rotating body

In frame rotating with body $\rightarrow$ centrifugal force

$$f_c = m\,\Omega^2 s$$



Ω = angular velocity

$\underline{s} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ shortest distance from rotation axis

$s = |\underline{s}|$ dist. from axis

θ = polar angle

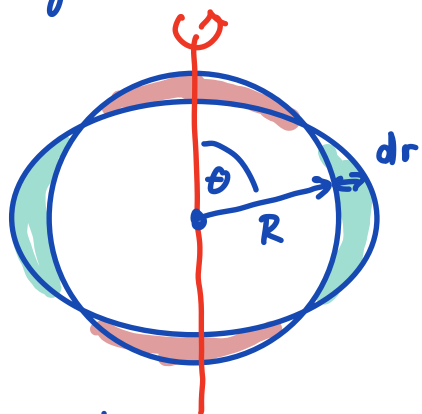
$\underline{s} = \underline{x} \sin\theta$

rotation adds "ficticious" acceleration & potential

centrifugal: $g_c = \Omega^2 \overset{R}{\underline{s}}$ $\quad \Phi_c = -\frac{1}{2}\Omega^2 s^2 \quad g_c = -\nabla\Phi_c$

gravitational: $g_G = \frac{GM}{R^2} \quad \Phi_G \approx |g|\,dr = \frac{GM}{R^2}\,dr$

note: Φ_G is linearized at surface



Total potential:

$$\Phi = \Phi_G + \Phi_C + \Delta\Phi$$

$\Delta\Phi$ = self potential

change of Φ_G due to deformation

Ignore $\Delta\Phi$

$$\Phi = \frac{MG}{R^2}\, dr(\theta) - \frac{1}{2}\frac{\Omega^2 R^2}{2}\sin^2\theta = \text{const.} + dr(\theta)$$

solve for dr

$$dr = \frac{R}{2}\, q \sin^2\theta + \text{const}$$

$$q = \frac{\Omega^2 R^3}{MG} = \frac{|g_c|}{|g_G|}$$

Constant determined by mass/volume conservation

$$\int_0^\pi dr(\theta)\, dS = 2\pi R^2 \int_0^\pi dr(\theta) \sin\theta\, d\theta = 0$$

$$\boxed{dr = dr_0\left(\sin^2\theta - \frac{2}{3}\right)} \qquad dr_0 = \frac{1}{2} R q$$

Earth: $R = 6371$ km $\quad g_s = 9.81\ \frac{m^2}{s} \quad \Omega = 7.6 \cdot 10^{-6}\ \frac{1}{s}$

$\Rightarrow q \simeq 3.5 \cdot 10^{-3} \qquad \Rightarrow dr_0 = 11$ km actual $dr_o = 21.4$ km

Error is due to self-potential $\Delta\Phi$

$$\Delta\Phi = -\frac{3}{5}\frac{\rho_1}{\rho_0} g_s \, dr(\theta)$$

where $\rho_0 = \frac{M}{V}$ and ρ_1 density of displaced material

$\rho_0 = 5500 \frac{kg}{m^3}$ $\rho_1 = 4500 \frac{kg}{m^3}$

$$\Rightarrow dr_0 = \frac{1}{2} R q \left(1 - \frac{3}{5}\frac{\rho_1}{\rho_0}\right)^{-1} \approx 21.9 \text{ km}$$