Lecture 16: Analysis of local deformation

Logistics: - HW 6 is due

(Last year we had 9 HW's)

- HW7 -> deformation

Last time: - Deformation map & gradient

$$\underline{x} = \varphi(\underline{X}) \qquad \underline{\underline{F}} = \nabla \varphi$$

$\underline{\underline{F}}$ is a natural measure of strain

locally $\boxed{\underline{x} = \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}}$

mapping of line segment: $d\underline{x} = \underline{\underline{F}}\, d\underline{X}$

volume changes: $dV_x = \underbrace{\det(\underline{\underline{F}})}_{J = \text{jacobian}}\, dV_X$

surface area changes: $\underline{n}\, dA_x = \boxed{J\, \underline{\underline{F}}^{-T} \underline{N}}\, dA_X$

Nanson's formula

Today: - Decompose $\underline{\underline{F}}$

Polar decomposition

$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$ $\qquad$ $\det(F) > 0 \Rightarrow$ admissible

right and left polar decomposition

$\underline{\underline{R}}$ = rotation (finite) $= \underline{\underline{Q}}$

$\left.\begin{array}{l} \underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}} \\ \underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} \end{array}\right\}$ sym. pos. definite

$\underline{v} \cdot \underline{\underline{S}}\,\underline{v} > 0$ for all $\underline{v} \in \mathcal{V}$

$\Rightarrow \lambda_i > 0 \qquad \lambda_i \in \mathbb{R}$

$\underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}} \quad \Rightarrow \quad \underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$

Show that $\underline{\underline{R}}$ is rotation:

1 $\quad \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}$ $\qquad$ orthonormal

2 $\quad \det(R) = 1$ $\qquad$ rotation

$(A^T)^{-1} = (A^{-1})^T \quad \Rightarrow \quad A = A^T$

$\underline{\underline{A}}^{-1} = (A^T)^{-1} = (A^{-1})^T$

$\Rightarrow \quad \underline{\underline{A}} = \underline{\underline{A}}^T \quad \Rightarrow \quad \underline{\underline{A}}^{-1} = (A^{-1})^T$

$\underline{\underline{U}} = \sqrt{F^TF} \Rightarrow U^2 = F^TF$

1 $\quad R$ is orthonormal

$$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-T}\underline{\underline{F}}^T\,\underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{U}}^{-1}\underline{\underline{U}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} = \underline{\underline{I}}$$

2) R is rotation

$$\det(\underline{\underline{R}}) = \det(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \det(\underline{\underline{F}})\det(\underline{\underline{U}}^{-1})$$
$$= \frac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$$

$\det(F) > 0$ $\qquad$ $\det(\underline{\underline{U}}) = \lambda_1 \lambda_2 \lambda_3 > 0$

Show $\underline{\underline{U}}$ is sym. pos. def (s.p.d.)

$|\underline{a}|^2 = \underline{a} \cdot \underline{a} > 0$ $\qquad$ $a = \underline{\underline{F}}\,\underline{v}$

$$\underline{\underline{F}}\,\underline{v} \cdot a = \underline{v} \cdot \underline{\underline{F}}^T \underline{a} = \underline{v} \cdot \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\underline{\underline{U}}^2 = \underline{\underline{C}}}\, \underline{v} > 0$$

$\Rightarrow$ $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ is spd $\qquad$ $\lambda_i > 0$

is $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$ also spd. ?

Tensor square root

$\underline{\underline{C}}$ spd $\qquad (\lambda_i, \underline{v}_i)$ $\qquad$ $\underline{\underline{C}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$

$$\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\; \underline{v}_i \otimes \underline{v}_i$$

eigenpair $\underline{\underline{U}}$ is $(\omega_i, \underline{v}_i)$ $\quad \omega_i = \sqrt{\lambda_i} > 0$

$\Rightarrow \quad \underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is spd

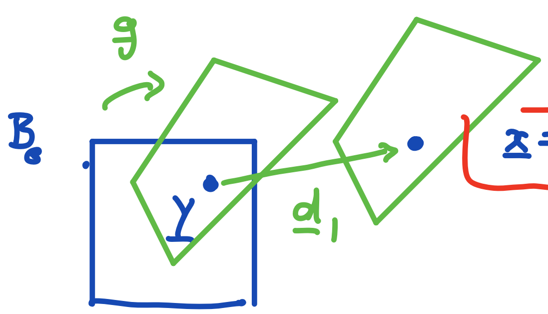
similarly $\quad \underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ is also s.p.d.

Analysis of local deformation

$$\boxed{\underline{x} = \varphi(\underline{X}) \approx \underline{c} + \underline{\underline{F}}\,\underline{X}} \qquad \underline{\underline{F}} = \nabla \varphi$$

To find strain tensor need to remove translations and rotations from deform.

1) Translation Fixed point decomposition



$d_i = d(\underline{X}) \qquad g = g(\underline{X})$

$$\boxed{\underline{x} = \varphi(\underline{X}) = \underline{d}_1 \circ \underline{g} = \underline{g} \circ \underline{d}_2}$$

$$= d_1(\underline{g}(\underline{X})) = \underline{g}(\underline{d}_2(\underline{X}))$$

where $\quad g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$

homogeneous deformation with fixed point $\underline{Y}$

$$y = g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$y = g(\underline{Y}) = \underline{Y} + \underline{\underline{F}}(\underline{Y} - \underline{Y}) = \underline{Y}$$

$$\underline{d}_i = \underline{X} + \underline{a}_i \qquad \text{translations}$$

write φ in terms of $\underline{Y}$

$$\underline{x} = \underline{c} + \underline{\underline{F}}\,\underline{X} \qquad \underline{y} = \underline{c} + \underline{\underline{F}}\,\underline{Y}$$

$$\underline{x} - \underline{y} = \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\boxed{\underline{x} = \varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})}$$

Taylor expansion around $\underline{Y}$

$$g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\underline{d}_1 = \underline{X} + \underline{a}_1$$

$$\underline{d}_1 \circ g = \underline{d}_1(g(\underline{X})) = g(\underline{X}) + \underline{a}_1$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{a}_1 \qquad \underline{a}_1 = \underline{y} - \underline{Y}$$

$$= \cancel{\underline{Y}} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \varphi(\underline{Y}) - \cancel{\underline{Y}} \qquad = \varphi(\underline{Y}) - \underline{Y}$$

$$= \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y}) = \varphi(\underline{X}) \quad \checkmark$$

$\Rightarrow$ always extract the translation and assume our deformation has a fixed point.

Stretch - Rotation decomposition

$\varphi(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X}-\underline{Y})$ fixed point $\underline{Y}$

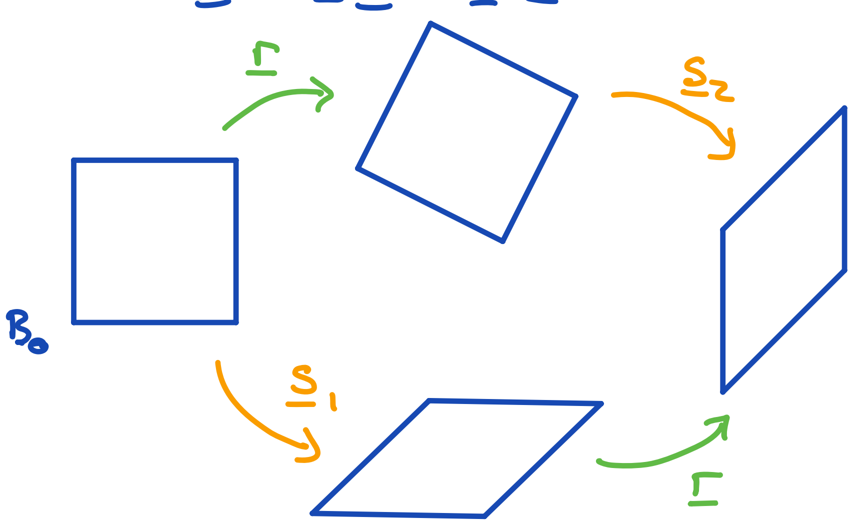
$$\boxed{\varphi(\underline{X}) = \underline{r}(\underline{X}) \circ \underline{s}_1(\underline{X}) = \underline{s}_2(\underline{X}) \circ \underline{r}(\underline{X})}$$

$\underline{r} = \underline{Y} + \underline{\underline{R}}(\underline{X}-\underline{Y})$ is rotation around $\underline{Y}$

$$\left.\begin{aligned} \underline{s}_1 &= \underline{Y} + \underline{\underline{U}}(\underline{X}-\underline{Y}) \\ \underline{s}_2 &= \underline{Y} + \underline{\underline{V}}(\underline{X}-\underline{Y}) \end{aligned}\right\} \text{ stretches from } \underline{Y}$$

where $\underline{\underline{R}}, \underline{\underline{U}}, \underline{\underline{V}}$ are defined by Polar decomp.

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$



$$\underline{\varphi} = \underline{r} \circ \underline{s}_1 = \underline{r}(\underline{s}_1(\underline{X})) = \underline{y} + \underline{\underline{R}}(\underline{s}_1(X) - \underline{y})$$
$$= \underline{y} + \underline{\underline{R}}(\cancel{\underline{y}} + \underline{\underline{U}}(\underline{x} - \underline{y}) - \cancel{\underline{y}})$$
$$= \underline{y} + \underline{\underline{R}}\,\underline{\underline{U}}\,(\underline{x} - \underline{y})$$
$$= \underline{y} + \underline{\underline{F}}(\underline{x} - \underline{y}) = \varphi$$

Stretch tensors

$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ $\quad$ $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ $\quad$ are s.p.d.

$\Rightarrow$ spectral decomposition

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i\, \underline{u}_i \otimes \underline{u}_i \quad \text{and} \quad \underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i\, \underline{v}_i \otimes \underline{v}_i$$

$(\lambda_i, \underline{u}_i)$ $\quad$ eigenpairs $\quad$ $(\lambda_i, \underline{v}_i)$

$\underline{\underline{U}}$ & $\underline{\underline{V}}$ have same λ's but different eigenvectors

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}} \qquad \underline{\underline{R}}^T\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{R}}^T\underline{\underline{V}}\,\underline{\underline{R}} \qquad \underline{\underline{U}} = \underline{\underline{R}}^T\underline{\underline{V}}\,\underline{\underline{R}}$$

$\underline{\underline{U}}$ & $\underline{\underline{V}}$ are related by $\underline{\underline{R}}$

$$p_U(\lambda) = \det(\underline{\underline{U}} - \lambda\underline{\underline{I}}) = \det(\underline{\underline{R}}^T\underline{\underline{V}}\,\underline{\underline{R}} - \lambda\,\underline{\underline{R}}^T\underline{\underline{R}})$$
$$= \det(\underline{\underline{R}}^T(\underline{\underline{V}} - \lambda\underline{\underline{I}})\underline{\underline{R}})$$

$$= \cancel{\det(\underline{\underline{R}}^T)} \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) \cancel{\det(\underline{\underline{R}})}$$

$$p_u(\lambda) = \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) = p_v(\lambda)$$

$\Rightarrow$ U & V have same eigenvalues

λ_i's are principal streches

$\underline{u}_i$ and $\underline{v}_i$ are left and right principal dir.

Relation between $\underline{u}_i$ & $\underline{v}_i$?

$$\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{u}_i \qquad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$

$$\underline{\underline{R}}\,\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{\underline{R}}\,\underline{u}_i$$

$$\underline{\underline{V}}\,\underbrace{\underline{\underline{R}}\,\underline{u}_i}_{\underline{v}_i} = \lambda \underbrace{\underline{\underline{R}}\,\underline{u}_i} \qquad \underline{\underline{V}}\,\underline{v}_i = \lambda_i \underline{v}_i$$

$$\boxed{\underline{v}_i = \underline{\underline{R}}\,\underline{u}_i}$$

In summary:

Any hom. def. f can be decomposed into a sequence of 3 elementary deformations:

1) Translation

2) Rotation around fixed point

3) Streck from fixed point

Example: $\varphi = \underline{s}_2 \circ \underline{r} \circ \underline{d}_2$

$\varphi = \underline{r} \circ \underline{s}_1 \circ \underline{d}_2$

....

These results are for hom. def ($\underline{\underline{F}} = \text{const}$)
but they apply to any def. in small neighborhood
by Taylor expansion.

Cauchy-Green Strain Tensor

For $\varphi(X)$ with $\nabla\varphi = \underline{\underline{F}}$

$$\boxed{\underline{\underline{C}} = F^T F}$$

right Cauchy-Green strain tens.
always s.p.d.

$\underline{\underline{F}}$ has information about both rotation and
streck, $\underline{\underline{C}}$ only contains streches

We use $\underline{\underline{C}}$ rather than $\underline{\underline{U}}$ to avoid square root!

$$\underline{\underline{U}} = \Sigma \lambda_i \; \underline{u}_i \otimes \underline{u}_i$$

$$\underline{\underline{C}} = \Sigma \lambda_i^2 \; \underline{u}_i \otimes \underline{u}_i$$

$\mu_i = \lambda_i^2$ eigs of $\underline{\underline{C}}$ are squares of principal stretches

$\underline{\underline{C}}$ is considered a "material strain tensor"

$$\underline{x} = \underline{\underline{F}}\,\underline{X} \qquad x_i = F_{iJ} X_J$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} \qquad C_{IJ} = F_{Jk} F_{kI}$$

Other strain tensors

I $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ Green-Lagrange strain tens.

$\Rightarrow$ linear theory

II $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T$ left Cauchy-Green strain tensor

III $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1})$ Euler Almansi strain tensor