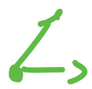
Lecture 17: Cauchy-Green Strain tensor

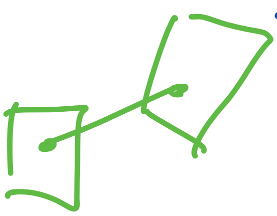
Logistics:
- HW5 all submitted
- HW6 3/7
- HW7 is due next week !

Last time:
- Analysis of local deformation
- Polar decomposition: $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$



  $\underline{\underline{R}}$ = rotation $\underline{\underline{U}}\,\underline{\underline{V}}$ = streches

- Decompose $\underline{\Phi}$ $\underline{\underline{U}} = \sqrt{F^T F}$



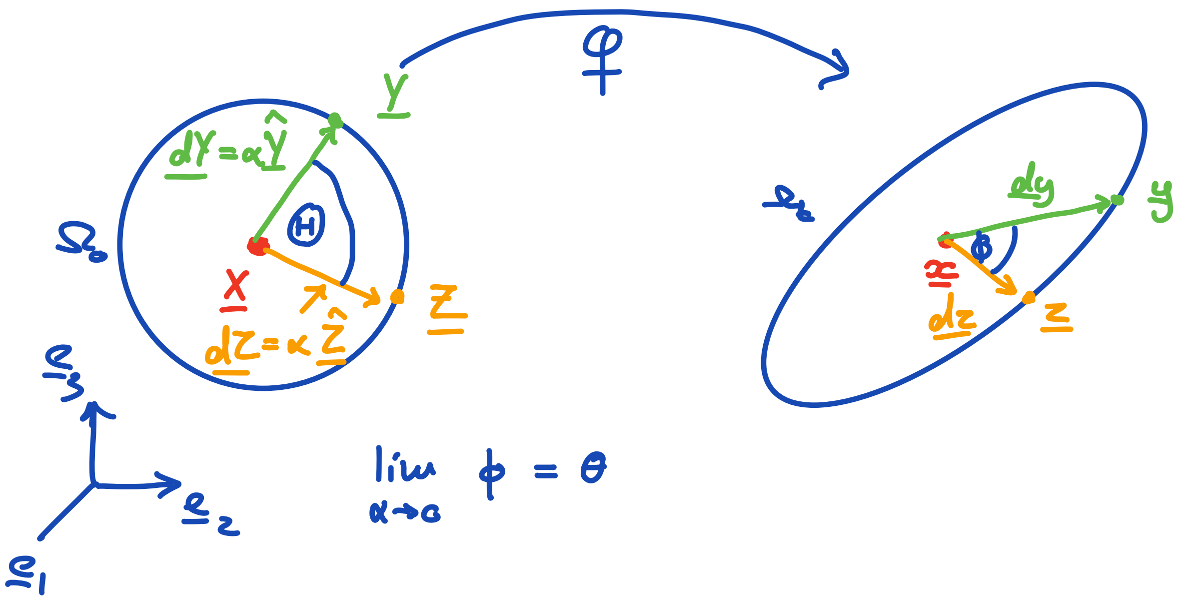
  1) Translation + Deformation with fixed point
  2) Rotation around fixed point R
  3) strech from fixed point $\underline{\underline{U}}$ $\underline{\underline{V}}$

- Cauchy-Green strain tenser: $\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{U}}^2}$

Today:
- Cauchy Green strain relations
- Interpretation of $\underline{\underline{C}}$
- Example
- Infinitesimal strain

Interpretation of $\underline{\underline{C}}$

Q: How are changes in relative position and orientation quantified by $\underline{\underline{C}}$?



$$\lim_{\alpha \to 0} \phi = \theta$$

Cauchy-Green strain relations

$$\boxed{\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}}$$

stretch of dir $\hat{\underline{Y}}$

$$\boxed{\cos\Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}}}$$

related to shear

I. Streches

$\lambda(\hat{\underline{Y}}) = \frac{|\underline{y}-\underline{x}|}{|\underline{Y}-\underline{X}|} = \frac{|d\underline{y}|}{|d\underline{Y}|}$ $\quad \lambda(\hat{\underline{Z}}) = \frac{|\underline{z}-\underline{x}|}{|\underline{Z}-\underline{X}|} = \frac{|d\underline{z}|}{|d\underline{Z}|}$ $\quad \lambda = \frac{\ell}{L}$

ratio of deformed to initial length

$d\underline{Y} = \alpha \hat{\underline{Y}}$

$$|d\underline{y}|^2 = d\underline{y} \cdot d\underline{y} = \underline{\underline{F}}\, d\underline{Y} \cdot \underline{\underline{F}}\, d\underline{Y} = d\underline{Y} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\, d\underline{Y} = d\underline{Y} \cdot \underline{\underline{C}}\, d\underline{Y}$$

$$= \alpha^2 \, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}$$

$$|d\underline{Y}|^2 = \alpha^2$$

$$\lambda^2 = \frac{|d\underline{y}|^2}{|d\underline{Y}|^2} = \frac{\alpha^2 \, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}{\alpha^2} = \lambda^2 = \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}$$

$$\Rightarrow \quad \lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}$$

✓

If $\underline{u}_i$ is right-principal strech

$$(\underline{\underline{C}} - \lambda^2 \underline{\underline{I}})\, \underline{u}_i = 0 \qquad \text{(no sum)}$$

$$\underline{u}_i \cdot \underline{\underline{C}}\, \underline{u}_i - \lambda_i^2 \, \cancel{\underline{u}_i \cdot \underline{u}_i} = 0$$

$$\underline{u}_i \cdot \underline{\underline{C}}\, \underline{u}_i = \lambda_i^2 \qquad \lambda_i = \sqrt{\underline{u}_i \cdot \underline{\underline{C}}\, \underline{u}_i}$$

$\Rightarrow$ λ_i represent streches in $\underline{u}_i$ direction

$\Rightarrow$ show that $\underline{u}_i$'s represent extremal values.

II Shear

Change in angle between two material lines

$$\boxed{\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})}$$

$$\cos\phi = \frac{\underline{dy} \cdot \underline{dz}}{|\underline{dy}|\,|\underline{dz}|}$$

$$\underline{dy} \cdot \underline{dz} = (\underline{\underline{F}}\,\underline{dY}) \cdot (\underline{\underline{F}}\,\underline{dZ}) = \underline{dY} \cdot \underline{\underline{C}}\ \underline{dZ} = \alpha^2\ \hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}$$

with: $|\underline{dy}| = \alpha\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}$ & $|\underline{dz}| = \alpha\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}$

substitute

$$\cos\phi = \frac{\alpha^2\ \hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}{\alpha\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}\ \alpha\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}} \xrightarrow{\alpha \to 0} \cos\theta$$

Components of $\underline{\underline{C}}$

$\{\underline{e}_i\}$

$$\boxed{\begin{aligned} C_{ii} &= \lambda^2(\underline{e}_i) \\ C_{ij} &= \lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\ \sin\gamma(\underline{e}_i, \underline{e}_j) \end{aligned}}$$

- diagonal comp. are squares of stretches in coord. dir.

- off-diag. comp. are related to shear between coord. dir.

For tensor: $\underline{A} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$A_{ij} = \underline{e}_i \cdot \underline{\underline{A}}\, \underline{e}_j \leftarrow$

Diagonal components:

$C_{ii} = \underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_i$ (no sum)

$\to$ 1st Cauchy Green: $\lambda(\underline{v}) = \sqrt{\underline{v} \cdot \underline{\underline{C}}\, \underline{v}}$

$\lambda^2(\underline{e}_i) = \underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_i = C_{ii}$

Off diagonal components:

$C_{ij} = \underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_j$

2nd Cauchy-Green: $\cos\theta(\underline{e}_i, \underline{e}_j) = \dfrac{\underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_j}{\sqrt{\underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_i}\sqrt{\underline{e}_j \cdot \underline{\underline{C}}\, \underline{e}_j}}$

Shear between $\underline{e}_i$ & $\underline{e}_j$ $(i \neq j)$:

$\gamma(\underline{e}_i, \underline{e}_j) = \frac{\pi}{2} - \theta(\underline{e}_i, \underline{e}_j)$

$\theta(\underline{e}_i, \underline{e}_j) = \frac{\pi}{2} - \gamma(\underline{e}_i, \underline{e}_j)$

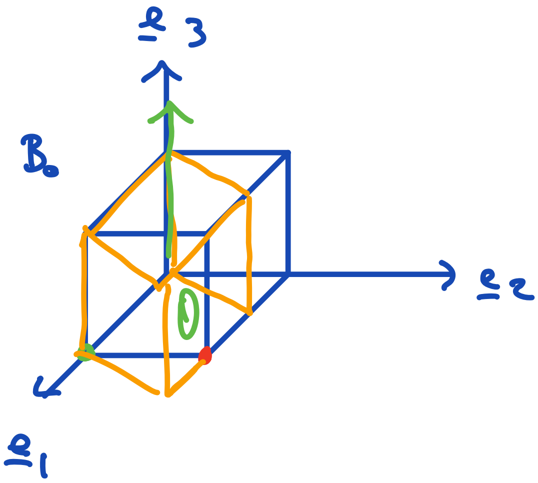
take cos C_{ij}

$$\frac{\underline{e}_i \cdot \underline{\underline{C}}\, \underline{e}_j}{\lambda(\underline{e}_i)\ \lambda(\underline{e}_j)} = \cos\left(\frac{\pi}{2} - \gamma(\underline{e}_i, \underline{e}_j)\right)$$

$$\Rightarrow \qquad C_{ij} = \lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\ \sin\left(\gamma(\underline{e}_i, \underline{e}_j)\right) \ \checkmark$$

$\alpha = 1$

Example: Simple Shear



$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} X_1 + \alpha X_2 \\ X_2 \\ X_3 \end{bmatrix} =$$

$$\underline{X}^* = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{x}^* = \begin{bmatrix} 1 + 1 \\ 1 \\ 0 \end{bmatrix}$$

Def. grad.:

$\frac{\partial \varphi_1}{\partial X_2}$

$$\underline{\underline{F}} = \nabla \underline{\varphi} = \frac{\partial \varphi_i}{\partial X_j}\, \underline{e}_i \otimes \underline{e}_j = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{const.}$$

$\frac{\partial \varphi_2}{\partial X_1}$

$\Rightarrow$ hom. def.

Cauchy - Green Strain.

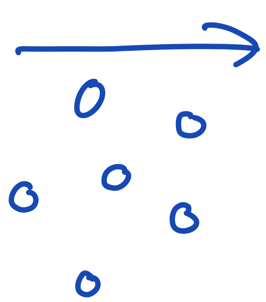
$$\underset{=}{C} = \underset{=}{F}^T \underset{=}{F} = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1+\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

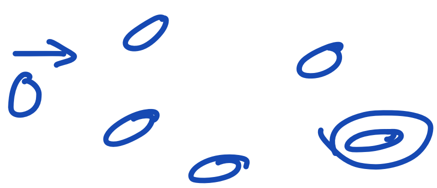
Shear between $\underline{e}_1$ and $\underline{e}_2$

$$\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \theta(\underline{e}_i, \underline{e}_2)$$

$$\cos\theta = \frac{\overbrace{\underline{e}_1 \cdot \underset{=}{C}\, \underline{e}_2}^{C_{12}}}{\sqrt{\underbrace{\underline{e}_1 \cdot \underset{=}{C}\, \underline{e}_1}_{C_{11}}}\ \sqrt{\underline{e}_2 \cdot \underset{=}{C}\, \underline{e}_2}} \quad C_{22} \qquad = \frac{\alpha}{\sqrt{1}\ \sqrt{1+\alpha^2}} =$$

$$\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \operatorname{acos}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$$





Shear between $\underline{e}_1$ & $\underline{e}_3$

$$\gamma(\underline{e}_1, \underline{e}_3) \qquad \cos\theta(\underline{e}_1, \underline{e}_3) = \frac{C_{13}}{\sqrt{C_{11}}\sqrt{C_{33}}} = \frac{0}{1 \cdot 1}$$

$$\gamma(\underline{e}_1, \underline{e}_3) = \frac{\pi}{2} - \underbrace{\mathrm{acos}\,0}_{\frac{\pi}{2}} = 0$$

What are extreme values of strech and their directions? $\rightarrow$ eigenvalue problem

$$(\underline{\underline{C}} - \lambda^2 \underline{\underline{I}})\,\underline{u} = 0$$

$$\begin{vmatrix} 1-\lambda^2 & \alpha & 0 \\ \alpha & 1+\alpha^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{vmatrix} = 0$$

$$\lambda_1^2 = 1 + \frac{\alpha^2}{2} + \alpha\sqrt{1+\frac{\alpha^2}{4}} \qquad \text{max strech}$$

$$\lambda_2^2 = 1$$

$$\lambda_3^2 = 1 + \frac{\alpha^2}{2} - \alpha\sqrt{1+\frac{\alpha^2}{4}} \qquad \text{min. strech}$$

Principal directions:

$$\underline{u}_1 = \left[\sqrt{1+\frac{\alpha^2}{4}} - \alpha/2,\ 1,\ 0\right]$$

$$\underline{u}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \underline{e}_3$$

$$\underline{u}_3 = \begin{bmatrix} \sqrt{1 + \frac{\alpha^2}{4}} + \frac{\alpha}{2}, & -1, & 0 \end{bmatrix}$$

$\Rightarrow$ no stretch in $\underline{e}_3$